§4.6, p.268, #21

This involves a sophisticated use of implicit differentiation. Without a graphing calculator the graph of $\sin y = xy$ will be hard to draw. But let us imagine that a graph was there, and that we want to find the slope and the concavity of that graph at a certain point, for example at $(0, \pi)$. By implicit differentiation, <u>regarding y as an implicit function of x</u> :

$$(\cos y)\, y' = y + x\, y' \qquad (1)$$

or

$$(\cos y - x)\, y' = y \qquad (2)$$

Hence

$$y' = y \,/\, (\cos y - x) \qquad (3)$$

In particular, $y'(0) = \pi / \cos\pi - 0 = -\pi$ is the slope of the tangent line at $(0, \pi)$.
The tangent line itself therefore has equation $y - \pi = -\pi(x - 0)$, or $y = \pi - \pi x$.
Differentiating (3) once more with respect to x gives

$$y'' = \frac{(\cos y - x)y' - y(-(\sin y)y' - 1)}{(\cos y - x)^2} \qquad (4)$$

using (2), (3) to simplify (4) one finally gets

$$y'' = \frac{2y(\cos y - x) + y^2 \sin y}{(\cos y - x)^3}$$

At $x = 0$, $y = \pi$, one has

$$y''(0) = \frac{2\pi(\cos\pi - 0) + \pi^2 \sin\pi}{(\cos\pi - 0)^3} = 2\pi$$

Since $2\pi > 0$ the graph of $\sin y = xy$ at $(0, \pi)$ is concave up. In the vicinity of $(0, \pi)$ the graph should be drawn above the line $y = \pi - \pi x$.

§ 4.6, p.268, #85

Given $z = g(y)$, $y = f(x)$. So $z = g(f(x))$, and by the chain rule $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$. Applying D_x to both sides, we get

$$\frac{d^2z}{dx^2} = \boxed{\frac{d}{dx}\left(\frac{dz}{dy}\right)} \cdot \frac{dy}{dx} + \frac{dz}{dy} \cdot \frac{d^2y}{dx^2} \qquad (1)$$

Now

$$\boxed{\frac{d}{dx}\left(\frac{dz}{dy}\right)} = \frac{d}{dy}\left(\frac{dz}{dy}\right) \cdot \frac{dy}{dx} = \frac{d^2z}{dy^2} \cdot \frac{dy}{dx}$$

. Substituting this into the box in (1) will give the final answer.

§ 10.4, p.715, #28

By standard formula

$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \cdots\cdots$

substituting x by 2x gives

$$\cos 2x = 1 - \frac{4x^2}{2\,!} + \frac{16x^4}{4\,!} - \frac{64x^6}{6\,!} + \cdots\cdots \qquad (1)$$

If we multiply (1) by $-\frac{1}{2}$ and then add $\frac{1}{2}$ to the result,

$$\text{we get } \frac{1}{2} - \frac{1}{2}\cos 2x = 0 - \frac{2x^2}{2\,!} + \frac{8x^4}{4\,!} - \frac{32x^6}{6\,!} + \cdots\cdots \qquad (2)$$

The left hand side equals $f(x) = \sin^2 x$, so (2) is the Maclaurin series for $\sin^2 x$.

§ 10.4, p.715, #29

Let $f(x) = \ell n(1+x)$. Repeated differentiation gives

$f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x^2)}$, $f'''(x) = \frac{2\,!}{(1+x)^3}$, $f^{(iv)}(x) = \frac{-3\,!}{(1+x)^4}$,

$\cdots\cdots$ and in general $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)\,!}{(1+x)^n}$

It follows that $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2\,!$,

$\cdots\cdots\ f^{(n)}(00 = (-1)^{n-1}(n-1)\,!$

The Taylor formula for x near a = 0 is

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2\,!}x^2 + \cdots\cdots + \frac{f^{(n)}(0)}{n\,!}x^n + \cdots\cdots$$

$$\text{So } \ell n(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} \pm \cdots\cdots + \frac{f^{(n)}(0)\,!}{n\,!}x^n \pm \cdots\cdots$$

Note that the coefficient of x^n simplifies to $\frac{(-1)^{n-1}}{n}$ [Answer]

§ 10.4, Extra question 1

To get the Maclaurin series for $x^3 \ell n(1+x)$, just multiply x^3 to the answer of p.175, # 29.