<u>Math 100 section 104 (LAM)</u> **Oct 18, 2004**
<u>Additional Course Material</u>

Two important (and famous) limits established by the use of logarithm.

(I) $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

<u>Proof</u> Let $A_n = \left(1+\frac{1}{n}\right)^n$. The strategy is to investigate first the limit of $\ell n\, A_n$ as $n \to \infty$. By the laws of logarithm

$$\lim_{n\to\infty} \ell n\, A_n = \lim_{n\to\infty} n\, \ell n\,(1+\tfrac{1}{n}) = \lim_{n\to\infty} \frac{\ell n(1+\frac{1}{n})}{1/n}$$

$$= \lim_{h\to\infty} \frac{\ell n(1+h) - \ell n(1)}{h} = \ell n'(1) = 1.$$

where we have set $1/n = h$ and used $\ell n\,(1) = 0$.

The limit, by the very definition of derivative, is $\ell n'(1)$, value of the derivative of the $\ell n\,(x)$ function at $x = 1$. Since $\ell n'(x) = (\ell n\, x)' = \frac{1}{x}$, that value is indeed $\frac{1}{1} = 1$.

We now return to A_n by exponentiating $\ell n\, A_n$:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} e^{\ell n\, A_n} = e^1 = e.$$

That the last limit equals e^1 is because as $n \to \infty$, $\ell n\, A_n \to 1$ and e^t is a continuous function at $t = 1$.

<u>NOTE</u> The above proof is an excellent example showing the interplay between limit, continuity, differentiability and inverse functions.

<u>Suggested numerical experimentation</u> : Get a close approximation to e by putting $n = 2{,}000{,}000$ into $(1 + \frac{1}{n})^n$, with the use of a pocket calculator.

(II). $\lim_{t\to\infty} \frac{e^t}{t^m} = \infty$ for any fixed positive integer m.

<u>Proof</u> Note $\ell n\,(\frac{e^t}{t^m}) = \ell n\,(e^t) - \ell n\,(t^m) = t - m\,(\ell n\, t)$.

As in the discussion of the limit in part **(I)**, it suffices to prove that

$$\lim_{t\to\infty} [\, t - m\,(\ell n\, t)\,] = \infty \qquad \textbf{(III)}$$

Let the real number t be expressed as a decimal with k digits occurring before the decimal point. This means that $10^k \geq t \geq 10^{k-1}$.

Taking ℓn gives $k(\ell n 10) \geq \ell n\, t \geq (k-1)(\ell n 10)$, or $kC \geq \ell n\, t \geq (k-1)C$ if we write $C = \ell n\, 10 \approx 2.30258 \ldots$. Now

$$t - m\,(\ell n\, t) \geq 10^{k-1} - m\,(\ell n\, t) \geq 10^{k-1} - m\,kC, \quad \text{or} \quad t - m\,(\ell n\, t) \geq k^2 - m\,kC$$

if we remember that $10^{k-1} \geq k^2$ for $k = 0,1,2,3, \ldots$. (The proof of this is left to you).

As $t \to \infty$, t has more and more integer digits, so $k \to \infty$ as well. It follows that

$$\lim_{t\to\infty} [\, t - m\,(\ell n\, t)] \geq \lim_{k\to\infty} k\,(k - mC) = \infty,$$

which establishes **(III)** as well as (**II**).