1. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 2$, $a_2 = 1$, and
   
   $$a_{n+1} = a_n + 6a_{n-1}$$

   for $n \geq 2$. Prove that, for all $n$, $a_n = 3^{n-1} + (-2)^{n-1}$.

2. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$, $a_2 = 2$, and
   
   $$a_{n+1} = a_n + a_{n-1} + 1$$

   for all $n \geq 2$. Conjecture a formula for $a_n$ and then prove your formula.

3. For any positive integer, $n$ is called prime if $n \geq 2$ and there exist no integers $a$
   such that $1 < a < n$ and $a|n$. Prime numbers are usually denoted by the letter $p$.

   Prove that any integer $n \geq 2$ is either prime or can be written as a product of
   (not necessarily distinct) primes.

4. (Binary Representation) Prove that any positive integer $n$ can be written as
   
   $$n = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$$

   for some integers $i_1, \ldots, i_k$ with the property that $0 \leq i_1 < i_2 < \cdots < i_k$. (You
   may assume the fact that for any positive integer $n$, there is a unique greatest
   integer $i$ such that $2^i \leq n$.)

   Can you prove this representation is unique?