1. Prove that if \( n \geq 2 \) is a natural number and \( A_1, A_2, \ldots, A_n \) are sets, then
\[
A_1 \cup A_2 \cup \cdots \cup A_n = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}
\]

**Proof.** We use induction on \( n \).
- **Base case:** \( n = 2 \). This is just De Morgan’s Law for sets.
- **Inductive step:** \( P(n) \implies P(n+1) \). Suppose that, for any \( n \) sets, the complement of the union is the intersection of their complements. Now suppose we have \( n+1 \) sets \( A_1, A_2, \ldots, A_{n+1} \). Then
\[
\begin{align*}
A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} &= (A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1} \\
&= (A_1 \cup A_2 \cup \cdots \cup A_n) \cap \overline{A_{n+1}} \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n) \cap \overline{A_{n+1}} \\
&= A_1 \cap A_2 \cap \cdots \cap A_n \cap \overline{A_{n+1}}
\end{align*}
\]

The second step holds by \( P(2) \) and the third step holds by \( P(n) \).

2. Prove that for every integer \( n \geq 5 \), \( 2^n > n^2 \).

**Proof.** We use induction on \( n \).
- **Base case:** \( n = 5 \). This is the statement that \( 2^5 > 5^2 \), i.e. \( 32 > 25 \).
- **Inductive step:** \( P(n) \implies P(n+1) \). Suppose that \( 2^n > n^2 \). We want to show that \( 2^{n+1} > (n+1)^2 \). It is sufficient for us to show that \( 2^{n+1} - 2^n > (n+1)^2 - n^2 \), i.e. to show that \( 2^n > 2n + 1 \). This is true because
\[
2^n > n^2 > 3n > 2n + 1
\]

The first inequality is by the inductive hypothesis, and the second inequality is because \( n > 3 \), and the third inequality because \( n > 1 \). Therefore, \( 2^{n+1} > (n+1)^2 \) and this completes the induction.

**Note:** In the inductive step, we showed that the left side increases more than the right side. We could have instead proved that \( \frac{2^{n+1}}{2^n} > \frac{(n+1)^2}{n^2} \). Simplifying both sides, this becomes \( 2 > (1 + \frac{1}{n})^2 \) - this is true for \( n = 5 \), and it therefore is true for all larger \( n \) as well.

3. Prove that for every positive odd integer \( n \), \( 5 | 4^n + 1 \).
Proof. Any odd integer $n$ can be written in the form $n = 2k + 1$ for some integer $k$. If $n$ is positive, then this means $k \geq 0$. So it is equivalent to prove that for every integer $k \geq 0$, $5|4^{2k+1} + 1$. We prove this by induction on $k$.

- Base Case: $k = 0$. $4^1 + 1 = 5$ so the divisibility clearly holds.
- Inductive step: Suppose that $5|4^{2k+1} + 1$. We wish to show that $5|4^{2(k+1)+1} + 1$, that is, $5|4^{2k+3} + 1$. By the inductive hypothesis, $4^{2k+1} \equiv -1 \pmod{5}$, and therefore

$$4^{2k+3} = 4^{2k+1} \cdot 4^2 \equiv (-1) \cdot 1 \equiv -1 \pmod{5}$$

and therefore, $5|4^{2k+3} + 1$, which completes the induction. □

4. Prove that every natural number $n \geq 8$ can be written in the form $n = 5a + 3b$ where $a, b$ are nonnegative integers.

**Proof 1.** Let the described property be denoted by $P(n)$. We will show that $P(8), P(9),$ and $P(10)$ are all true, and then prove that $P(n) \implies P(n+3)$ for all $n \geq 8$. This will separately show that $P(8 + 3k), P(9 + 3k),$ and $P(10 + 3k)$ hold for all nonnegative integers $k$, thereby proving $P(n)$ for all $n \geq 8$.

- Base case: $n = 8, 9, 10$. $8 = 5(1) + 3(1), 9 = 5(0) + 3(3), 10 = 5(2) + 3(0)$.
- Inductive step: Suppose that $n = 5a + 3b$ for some nonnegative integers $a, b$. Then $n + 3 = 5a + 3(b + 1)$. $b + 1$ is also a nonnegative integer, so this implies $P(n + 3)$. This completes the induction. □

**Proof 2.** Let $P(n)$ denote the property described above, and let $Q(n) = P(n) \land P(n + 1) \land P(n + 2)$. That is, $Q(n)$ is the property that $n, n + 1, n + 2$ can all be written in the given form. We will prove $\forall n \geq 8, Q(n)$ by induction on $n$.

- Base case: $n = 8, 9, 10$. $8 = 5(1) + 3(1), 9 = 5(0) + 3(3), 10 = 5(2) + 3(0)$.
- Inductive step: Suppose that $Q(n)$ is true: namely, that $n = 5a + 3b, n + 1 = 5c + 3d, n + 2 = 5e + 3f$. We must show that $n + 1, n + 2,$ and $n + 3$ have the required property $P$. By assumption, $n + 1 = 5c + 3d, n + 2 = 5e + 3f,$ and $n + 3 = 5a + 3(b + 1)$. Thus, $Q(n + 1)$ is true, which completes the induction. □

5. Consider the following statement.

For every integer $k \geq 5$, there exists a natural number $N$ such that for every integer $n \geq N$, $2^n > n^k$.

(a) What is wrong with the following argument disproving the statement?

**Proof.** Suppose there is such an $N$. Then for every $k \geq 5$ and every $n \geq N$, we have $2^n > n^k$. This means $\log_n(2^n) > k$. But this doesn’t hold when $k \geq \log_n(2^n)$. This is a contradiction! □
Solution: The statement is saying that $N$ can be chosen dependent on $k$, whereas the proof is assuming that $N$ is chosen independent of $k$. The reason $N$ is dependent on $k$, is that the part quantifying $N$ is inside the sentence quantified by $k$.

$$\forall k \geq 5, (\exists N \in \mathbb{N}, (\forall n \geq N, (2^n > n^k)))$$

(b) I think there is a better way to name the variables:

For every integer $k \geq 5$, there exists a natural number $N_k$ such that for every integer $n \geq N_k$, $2^n > n^k$.

Why is this better? This reminds us that the value of $N$ that is chosen is dependent on what $k$ is.

(c) Can you prove the statement? (This is challenging and may require multiple steps! Hint: base case $N_k = 2^k$.)

Proof. For each $k \geq 5$, let $N_k = 2^k$. We prove that, for all $n \geq 2^k$, $2^n > n^k$ by induction on $n$.

• Base case: $n = 2^k$. We must check that, for each $k \geq 5$, $2^{(2^k)} > (2^k)^k$. But $(2^k)^k = 2^{(k^2)}$. The inequality $2^{(2^k)} > 2^{(k^2)}$ for $k \geq 5$ now follows because we proved in Q2 that $2^k > k^2$ for $k \geq 5$.

• Inductive step: Suppose that $2^n > n^k$. We must show that $2^{n+1} > (n+1)^k$.

It is sufficient to show that $2^{n+1} > (n+1)^k$, as then we have that

$$2^{n+1} = 2^n \cdot \frac{2^{n+1}}{2^n} > 2^n \cdot \frac{(n+1)^k}{n^k} = \frac{2^n}{n^k} \cdot (n+1)^k > (n+1)^k$$

So let’s show that $\frac{2^{n+1}}{2^n} > \frac{(n+1)^k}{n^k}$. Simplifying both sides, it is equivalent to show that $2 > (1 + \frac{1}{n})^k$. Since $n \geq 2^k$, it is sufficient to show that $2 > (1 + \frac{1}{2^k})^k$. When we expand the binomial product on the right hand side, we get $2^k$ terms: one of them is equal to 1, and every other term is at most $\frac{1}{2}$. Therefore, the right hand side is at most $1 + \frac{2^k - 1}{2^k} = 2 - \frac{1}{2^k}$. Therefore, the inequality $2 > (1 + \frac{1}{2^k})^k$ holds, and this proves that the induction holds. \qed