Solutions to Math 105 Midterm 1 (Version I)

1. (a) \(2(x+2)-5(y-3)+1(z-1) = 0\)
   
   (or \(2x-5y+z = -18\))

   (b) \(\vec{w} = \langle 1, -3, 0 \rangle\), because

   \[\vec{u} \cdot \vec{w} = (3, 1, -2) \cdot (1, -3, 0) = 3 - 3 + 0 = 0.\]

   (Students can use any of the following vectors:
   \(\vec{w} = \langle a, 2b-3a, b \rangle\), \(a, b\) are constants and \(a \neq 0\))

2. (a) Since \((0, 2)\) is in the domain of \(f(x, y)\), we have

   \[4 - a^2 - 2^2 \geq 0 \Rightarrow -a^2 \geq 0 \Rightarrow a = 0.\]

   (b) \(f(x, y) = 0 \Rightarrow \frac{1}{1 + \sqrt{4-x^2-y^2}} = 2 \Rightarrow 1 + \sqrt{4-x^2-y^2} = \frac{1}{2}\)

   \[\Rightarrow \sqrt{4-x^2-y^2} = \frac{1}{2} - 1 \Rightarrow 4 - x^2 - y^2 = \left(\frac{1}{2} - 1\right)^2\]

   or \(x^2 + y^2 = 4 - \left(\frac{1}{2} - 1\right)^2\) \(= 4 - \left(\frac{1}{2}\right)^2 = 4 - \frac{1}{4} = \frac{15}{4}\)

   \(\Rightarrow x^2 + y^2 = \frac{15}{4}\)

   (1)

   If \(z_0 = \frac{1}{2}\), then (1) becomes \(x^2 + y^2 = 4 - (3-1)^2 = 0 \Rightarrow (x, y) = (0, 0)\)

   If \(z_0 = \frac{1}{2}\), then (1) becomes \(x^2 + y^2 = 4 - (2-1)^2 = 3\), which is a circle
3. (a) \[ f_x = 10x \ln \left( 5x^2 + 4y^2 \right) + \left( 5x^2 + 4y^2 \right) \frac{1}{5x^2 + 4y^2} (10xy) \]

\[ = 10x \ln (5x^2 + 4y^2) + (5x^2 + 4y^2) \frac{1}{(5x^2 + 4y^2)} (10xy) \]

or \[ f_x = 10x \ln (5x^2 + 4y^2) + 10xy \]

(b) \[ f_{xy} = \frac{\partial}{\partial y} (f_x) = \frac{2}{y} \left( 10x \ln (5x^2 + 4y^2) + 10xy \right) \]

\[ = 10x \cdot \frac{5x + 8y}{5x^2 + 4y^2} \]
4(a) \( f_x = e^{xy} + xe^{xy}y = (1+xy)e^{xy} \)

\( f_y = xe^{xy}, \quad x = x^2 e^{xy} \)

The system
\[
\begin{align*}
\begin{cases}
 f_x = (1+xy)e^{xy} = 0 & (1) \\
 f_y = x^2 e^{xy} = 0 & (2)
\end{cases}
\end{align*}
\]

does not have a solution. In fact, (1) implies that
\( 1 + xy = 0 \) \( \Rightarrow (3) \)

and (2) implies that \( x = 0 \) or \( x = 0 \) \( \Rightarrow (4) \)

Using (3) and (4), we get \( 1 + 0 \cdot y = 0 \), which is impossible. Hence, \( f(x,y) \) does not have any critical point.

(b) \( f_x = 4x, \quad f_y = A + 6y \)

Since \( f_x(0, -1) = 0 \), we get \( 0 = A - 6 \Rightarrow A = 6 \).

Also, \( 6 = f(0, -1) = 6(-1) + 3(-1)^2 + B = -6 + 3 + B \Rightarrow B = 11 \).
5. (a) We need to solve the system
\[
\begin{align*}
   f_x &= 3x^2 - 6y - 9 = 0 \quad (1) \\
   f_y &= -6x + 6y = 0 \quad (2)
\end{align*}
\]
By (2), \( y = x \) \( (3) \). Using (3), (1) becomes
\[
3x^2 - 6x - 9 = 0 \Rightarrow x = -1 \text{ or } x = 3.
\]
Hence, all critical points are \((-1, -1)\) and \((3, 3)\).

(b) \( f_{xx} = 6x \), \( f_{yy} = 6 \), \( f_{xy} = -6 \).

<table>
<thead>
<tr>
<th>( f_{xx} )</th>
<th>( f_{yy} )</th>
<th>( f_{xy} )</th>
<th>( D = f_{xx} f_{yy} - f_{xy}^2 )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1, -1)</td>
<td>-6</td>
<td>6</td>
<td>-72</td>
<td>Saddle point</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>18</td>
<td>6</td>
<td>72</td>
<td>Local minimum</td>
</tr>
</tbody>
</table>
6. (a) Objective function is \( z = f(x, y) = x \)

Constraint is \( g(x, y) = x^2 + 6y^2 + 3xy = 40 = 0 \)

( or \( g(x, y) = x^2 + 6y^2 - 3xy = 40 \) )

We need to solve the system:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= g_x = 1 \\
\frac{\partial f}{\partial y} &= g_y = 0
\end{align*}
\]

By (1) and (2), \( x \neq 0 \). Hence, (2) becomes

\[ 12y + 3x = 0 \Rightarrow x = -4y \quad (4) \]

Using (4), (3) becomes \((-4y)^2 + 6y^2 + 3(-4y)y = 40 \) \( 2 \)

\[ y^2 = 4 = y = \pm 2 \]

\[ y = 2 \Rightarrow x = -4 \cdot 2 = -8 \]
\[ y = -2 \Rightarrow x = -4(-2) = 8 \]

Since \( f(-8, 2) = -8 \) and \( f(8, -2) = 8 \), we get that the minimum value of \( f(x, y) = x \) is \( 8 \), and the maximum value of \( f(x, y) = x \) is \( -8 \).

(b) By (a), the point is \( (8, -2) \).
5. (a) We need to solve the system
\[ \begin{align*}
  f_x &= 2x^2 + 2x = 0 \quad (1) \\
  f_y &= 6y^2 + x^2 + 10y = 0 \quad (2)
\end{align*} \]

From (1), we have \( 2x(2x + 1) = 0 \) so \( x = 0 \) or \( y = -1 \).

Case 1 \( x = 0 \), in which case, (2) gives \( 6y^2 + 10y = 0 \) \( \Rightarrow \)
\[ 2y(3y + 5) = 0 \Rightarrow \quad y = 0 \quad \text{or} \quad y = -\frac{5}{3}. \]
So \((0, 0)\) and \((0, -\frac{5}{3})\) are critical points. We get in Case 1.

Case 2 \( y = -1 \), in which case, (2) gives \( 6 + x^2 = 10 = 0 \) \( \Rightarrow \)
\[ x^2 = 4 \Rightarrow \quad x = 2 \quad \text{or} \quad x = -2. \]
Hence, we get two more critical points \((2, -1)\)
and \((-2, -1)\) in Case 2.

Thus, all critical points are \((0, 0)\), \((0, -\frac{5}{3})\), \((2, -1)\) and \((-2, -1)\).

(b) \( f_{xx} = 2y + 2 \), \( f_{yy} = 12y + 10 \), \( f_{xy} = 2x \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & f_{xx} & f_{yy} & f_{xy} & D = f_{xx}f_{yy} - f_{xy}^2 & \text{Conclusion} \\
\hline
(0, 0) & 2 & 10 & 0 & + & \text{local min} \\
\hline
(0, -\frac{5}{3}) & -\frac{4}{3} & -10 & 0 & + & \text{local max.} \\
\hline
(2, -1) & 0 & -2 & 4 & - & \text{saddle point} \\
\hline
(-2, -1) & 0 & -2 & -4 & - & \text{saddle point} \\
\hline
\end{array}
\]
(a) Objective function is \( z = f(x, y) = x \)

Constraint is \( g(x, y) = x^2 + 6y^2 + 3xy - 4 = 0 \)

( or \( g(x, y) = x^2 + 6y^2 + 3xy = 4 \) )

We need to solve the system:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \quad \text{or} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\
g(x, y) &= 0
\end{align*}
\]

By (1) and (2), \( \lambda \neq 0 \). Hence, (2) becomes

\[
12y + 3x = 0 \implies x = -4y 
\]

Using (4), (3) becomes \((-4y)^2 + 6y^2 + 3(-4y) y = 4 \) \( \implies 
\]

\[
y^2 = 4 \implies y = \pm 2.
\]

\( y = 2 \implies x = -4 \), \( y = -2 \implies x = -4 \).

Since \( f(-8, 2) = -8 \) and \( f(8, -2) = 8 \), we get that

the maximum value of \( f(x, y) = x \) is \( 8 \), and

the minimum value of \( f(x, y) = x \) is \( -8 \).

(b) By (a), the point is \((8, -2)\).