# Midterm Exam #2—Math 101, Section 201

March 13, 2015

Duration: 50 minutes

Solution

______________________ ______________________ ______________________

Surname (Last Name) Given Name Student Number

**Do not open this test until instructed to do so!** This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.



| Problem | Out of | Score | Problem | Out of | Score |
|---|---|---|---|---|---|
| 1 | 6 | | 5 | 8 | |
| 2 | 6 | | 6 | 8 | |
| 3 | 6 | | 7 | 8 | |
| 4 | 3 | | **Total** | 45 | |

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Evaluate $\int \sin^3 x \cos^2 x \, dx$.

$= \int \sin^2 x \cos^2 x \sin x \, dx = \int (1-\cos^2 x)\cos^2 x \sin x \, dx$ ①

$= \int (1-u^2)u^2(-du)$ ① $= \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$u = \cos x$

$du = -\sin x \, dx$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$ ①

1b. **[3 pts]** $\int (\ln t)\sqrt{t}\, dx.$ $= \frac{2}{3}t^{3/2}\ln t - \int \frac{1}{t}\cdot\frac{2}{3}t^{3/2}\,dt$ ①

$u = \ln t,\ v' = \sqrt{t}$ ①

$u' = \frac{1}{t},\ v = \frac{2}{3}t^{3/2}$

$= \frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int t^{1/2}\,dt = \frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\cdot\frac{2}{3}t^{3/2} + C$ ①

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** $\int \frac{\sqrt{x^2-9}}{x}\,dx.$

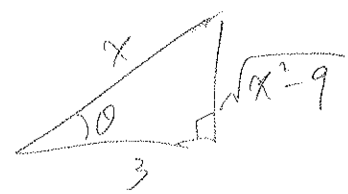
$$\underset{x=3\sec\theta}{=} \int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \quad 1$$

$$dx = 3\sec\theta\tan\theta\,d\theta$$

$$\left(0 \le \theta < \tfrac{\pi}{2} \text{ or } \pi \le \theta < \tfrac{3\pi}{2}\right)$$

$$= \int 3\sqrt{\tan^2\theta}\,\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)\,d\theta \quad 1$$

$$= 3\tan\theta - 3\theta + C = 3\cdot\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C \quad 1$$




2b. **[3 pts]** $\int \frac{-x+2}{2x^2+x-1}\,dx.$

$$\frac{-x+2}{2x^2+x-1} = \frac{-x+2}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \quad 1$$

$$-x+2 = A(x+1) + B(2x-1) \quad \tfrac{1}{2}$$

$x=-1 \Rightarrow \quad 3 = B(-3) \Rightarrow B = -1. \quad \tfrac{1}{2}$

$x=\tfrac{1}{2} \Rightarrow \quad -\tfrac{1}{2}+2 = A(\tfrac{1}{2}+1) \Rightarrow \tfrac{3}{2} = A(\tfrac{3}{2}) \Rightarrow A = 1.$

$$\int \frac{-x+2}{2x^2+x-1}\,dx = \int \frac{dx}{2x-1} + \int \frac{-dx}{x+1} = \frac{1}{2}\ln|2x-1| - \ln|x+1| + C \quad 1$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** $\int_0^\infty \frac{1}{(3x+1)^4}\,dx.$ $= \lim_{t\to\infty} \int_0^t \frac{1}{(3x+1)^4}\,dx$ ①

$$\underset{\substack{u=3x+1 \\ du=3dx}}{=} \lim_{t\to\infty} \int_1^{3t+1} \frac{1}{u^4}\left(\frac{du}{3}\right) ① = \lim_{t\to\infty} \frac{1}{3} \left.\frac{u^{-3}}{-3}\right|_1^{3t+1}$$

$$= -\frac{1}{9} \lim_{t\to\infty} \left(\frac{1}{(3t+1)^3} - 1\right) = \frac{1}{9}. ①$$

3b. **[3 pts]** Determine whether $\int_2^3 \frac{dx}{(x-2)^2}$ is convergent or divergent.

$$\int_2^3 \frac{dx}{(x-2)^2} = \lim_{t\to 2^+} \int_t^3 \frac{dx}{(x-2)^2} ① = \lim_{t\to 2^+} \left(\left.\frac{-1}{x-2}\right|_t^3\right)$$

$$= \lim_{t\to 2^+} \left(-1 + \frac{1}{t-2}\right) = -1 + \lim_{t\to 2^+} \frac{1}{t-2} = \infty. ①$$

Hence, $\int_2^3 \frac{dx}{(x-2)^2}$ diverges. ①

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Determine whether the sequence $a_n = \ln(3n^2 + n) - \ln(n^2 + 2n + 7)$ converges or diverges.

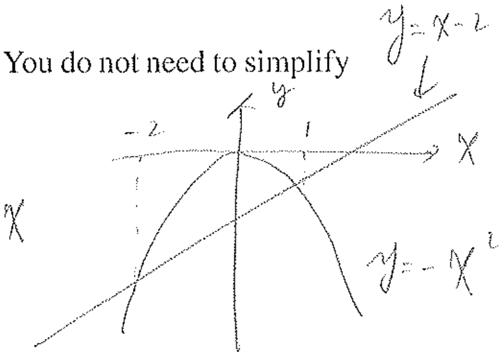
$$a_n = \ln \frac{3n^2+n}{n^2+2n+7} = \ln \frac{3+\frac{1}{n}}{1+\frac{2}{n}+\frac{7}{n^2}} \quad 1 \quad \to \ln 3 \quad 1$$

The sequence $\{a_n\}$ converges 1

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. Let $R$ be the region bounded by the line $y = x - 2$ and the parabola $y = -x^2$. It is known that the area of the region $R$ is $\frac{9}{2}$.

(a) **[4 pts]** Find the $x$-coordinate $\bar{x}$ of the centroid of the region $R$. You do not need to simplify your answer.




$x - 2 = -x^2 \Rightarrow x = -2$ or $x = 1$. 1

$$\bar{x} = \frac{1}{(9/2)} \int_{-2}^{1} x\left(-x^2 - (x-2)\right) dx \overset{1}{=} \frac{2}{9} \int_{-2}^{1} \left(-x^3 - x^2 + 2x\right) dx$$

$$= \frac{2}{9}\left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right)\Big|_{-2}^{1} \quad 1$$

$$= \frac{2}{9}\left(\left(-\frac{1}{4} - \frac{1}{3} + 1\right) - \left(-\frac{(-2)^4}{4} - \frac{(-2)^3}{3} + (-2)^2\right)\right) \quad 1$$

(b) **[4 pts]** Find the $y$-coordinate $\bar{y}$ of the centroid of the region $R$. You do not need to simplify your answer.

$$\bar{y} = \left(\frac{1}{(9/2)}\right) \int_{-2}^{1} \frac{1}{2}\left\{(-x^2)^2 - (x-2)^2\right\} dx \quad 1$$

$$= \frac{1}{9} \int_{-2}^{1} \left(x^4 - (x^2 - 4x + 4)\right) dx = \frac{1}{9} \int_{-2}^{1} \left(x^4 - x^2 + 4x - 4\right) dx \quad 1$$

$$= \frac{1}{9}\left(\frac{x^5}{5} - \frac{x^3}{3} + 2x^2 - 4x\right)\Big|_{-2}^{1} \quad 1$$

$$= \frac{1}{9}\left\{\left(\frac{1}{5} - \frac{1}{3} + 2 - 4\right) - \left(\frac{(-2)^5}{5} - \frac{(-2)^3}{3} + 2(-2)^2 - 4(-2)\right)\right\}. \quad 1$$

6. The graph of a function $y = f(x)$ is the curve $C$ that passes through the point $(0, -5)$ and whose slope at $(x, y)$ is $\frac{x}{2ye^{y^2}}$.

(a) **[6 pts]** Find an equation of the curve $C$.

$$\frac{dy}{dX} = \frac{X}{2ye^{y^2}} \text{ (1)} \Rightarrow 2ye^{y^2}dy = X\,dX \text{ (1)}$$

$$\int 2ye^{y^2}dy = \int X\,dX \Rightarrow e^{y^2} = \frac{X^2}{2} + C \text{ (1)}$$

Since $(0,-5)$ satisfies the equation above, we have $e^{(-5)^2} = \frac{0^2}{2} + C$ (1) $\Rightarrow C = e^{25}$. (1)

Hence, an equation of the curve $C$ is given

$$e^{y^2} = \frac{X^2}{2} + e^{25}. \text{ (1)} \qquad (1)$$

(b) **[2 pts]** Find $f(x)$ by solving the equation in the part (a) above.

By (1), $y^2 = \ln\left(\frac{X^2}{2} + e^{25}\right)$ (1)

Since $y(0) < 0$, $y = -\sqrt{\ln\left(\frac{X^2}{2} + e^{25}\right)} = f(X)$. (1)

7. **[8 pts]** Let $f(x)$ be a continuous function defined on $[3, 5]$. It is known that $0 \leq f'(x) \leq 2$ and $0 \leq f''(x) \leq 4$ for all $x$ in $[3, 5]$. Let $g(x) = xf(x)$. How large should we take $n$ in order to guarantee that the Trapezoidal Rule approximation for $\int_3^5 g(x)\,dx$ is accurate to within $0.25$? You may use the fact: *Suppose* $|f''(x)| \leq K$ *for* $a \leq x \leq b$. *If* $E_T$ *is the error in the Trapezoidal Rule,* for $\int_a^b f(x)dx$ *then* $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$.

Let $g(x) = xf(x) = xf$. Then

$g' = f + xf'$, ① $g'' = f' + f' + xf'' = 2f' + xf''$ ①

$|g''(x)| = 2f' + xf'' \leq 2\cdot 2 + 5\cdot 4 = 24 (= K)$ ②

Hence, It is enough to find $n$ such that

$$\frac{24\cdot(5-3)^3}{12n^2} \leq 0.25 = \frac{1}{4} \quad ①$$

or $\dfrac{2\cdot 2^3}{n^2} \leq \dfrac{1}{4} \Rightarrow n^2 > 2^4\cdot 2^2$ ①

$\Rightarrow n \geq \sqrt{2^4\cdot 2^2} = 2^2\cdot 2 = 8$ ①

Thus, we can take $\boxed{n = 8}$ ①