

Midterm Exam #2—Math 101, Section 201

March 13, 2015

Duration: 50 minutes

Solution

Surname (Last Name)

Given Name

Student Number

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (c) purposely viewing the written papers of other examination candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		Total	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. [3 pts] Evaluate $\int \sin^3 x \cos^2 x dx$.

$$\begin{aligned}
 &= \int \sin^2 x \cos^2 x \sin x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx \quad \textcircled{1} \\
 &= \int (1 - u^2) u^2 (-du) \quad \textcircled{1} = \int (u^2 - u^4) du = \frac{u^3}{3} - \frac{u^5}{5} + C \\
 u &= \cos x \\
 du &= -\sin x dx \\
 &= \frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + C \quad \textcircled{1}
 \end{aligned}$$

1b. [3 pts] $\int (\ln t) \sqrt{t} dt$.

$$\begin{aligned}
 &= \frac{2}{3} t^{3/2} \ln t - \int \frac{1}{t} \cdot \frac{2}{3} t^{3/2} dt \quad \textcircled{1} \\
 u &= \ln t, \quad v = \sqrt{t} \quad \textcircled{1} \\
 u' &= \frac{1}{t}, \quad v = \frac{2}{3} t^{3/2} \\
 &= \frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \int t^{1/2} dt = \frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \cdot \frac{2}{3} t^{3/2} + C \quad \textcircled{1}
 \end{aligned}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. [3 pts] $\int \frac{\sqrt{x^2-9}}{x} dx.$ $\underline{\hspace{2cm}}$ $\int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta} \cdot 3\sec\theta \tan\theta d\theta$ $\textcircled{1}$


$x = 3\sec\theta$

$dx = 3\sec\theta \tan\theta d\theta$

$(0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2})$

$= \int 3\sqrt{\tan^2\theta} \tan\theta d\theta = 3 \int \tan^2\theta d\theta = 3 \int (\sec^2\theta - 1) d\theta$ $\textcircled{1}$

$= 3 \tan\theta - 3\theta + C = 3 \cdot \frac{\sqrt{x^2-9}}{3} - 3 \sec^{-1}\left(\frac{x}{3}\right) + C$ $\textcircled{1}$



2b. [3 pts] $\int \frac{-x+2}{2x^2+x-1} dx.$

$$\frac{-x+2}{2x^2+x-1} = \frac{-x+2}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$
 $\textcircled{1}$

$$-x+2 = A(x+1) + B(2x-1)$$

$$x = -1 \Rightarrow 3 = B(-3) \Rightarrow B = -1.$$
 $\textcircled{1/2}$

$$x = \frac{1}{2} \Rightarrow -\frac{1}{2} + 2 = A\left(\frac{1}{2} + 1\right) \Rightarrow \frac{3}{2} = A\left(\frac{3}{2}\right) \Rightarrow A = 1.$$
 $\textcircled{1/2}$

$$\int \frac{-x+2}{2x^2+x-1} dx = \int \frac{dx}{2x-1} + \int \frac{-dx}{x+1} = \frac{1}{2} \ln|2x-1| - \ln|x+1| + C$$
 $\textcircled{1}$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. [3 pts] $\int_0^{\infty} \frac{1}{(3x+1)^4} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{(3x+1)^4} dx$ ①

$u = 3x+1$
 $du = 3dx$

$$\lim_{t \rightarrow \infty} \int_1^{3t+1} \frac{1}{u^4} \left(\frac{du}{3}\right) = \lim_{t \rightarrow \infty} \frac{1}{3} \frac{u^{-3}}{-3} \Big|_1^{3t+1}$$

$$= -\frac{1}{9} \lim_{t \rightarrow \infty} \left(\frac{1}{(3t+1)^3} - 1 \right) = \frac{1}{9}$$
 ①

3b. [3 pts] Determine whether $\int_2^3 \frac{dx}{(x-2)^2}$ is convergent or divergent.

$$\int_2^3 \frac{dx}{(x-2)^2} = \lim_{t \rightarrow 2^+} \int_t^3 \frac{dx}{(x-2)^2} = \lim_{t \rightarrow 2^+} \left(\frac{-1}{x-2} \Big|_t^3 \right)$$

$$= \lim_{t \rightarrow 2^+} \left(-1 + \frac{1}{t-2} \right) = -1 + \lim_{t \rightarrow 2^+} \frac{1}{t-2} = \infty$$
 ①

Hence, $\int_2^3 \frac{dx}{(x-2)^2}$ diverges. ①

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. [3 pts] Determine whether the sequence $a_n = \ln(3n^2 + n) - \ln(n^2 + 2n + 7)$ converges or diverges.

$$a_n = \ln \frac{3n^2 + n}{n^2 + 2n + 7} = \ln \frac{3 + \frac{1}{n}}{1 + \frac{2}{n} + \frac{7}{n^2}} \rightarrow \ln 3$$

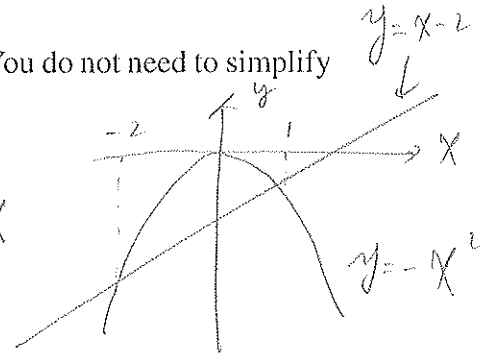
The sequence $\{a_n\}$ converges.

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. Let R be the region bounded by the line $y = x - 2$ and the parabola $y = -x^2$. It is known that the area of the region R is $\frac{9}{2}$.

(a) [4 pts] Find the x -coordinate \bar{x} of the centroid of the region R . You do not need to simplify your answer.

$$\begin{aligned} x-2 &= -x^2 \Rightarrow x = -2 \text{ or } x = 1. \quad \textcircled{1} \\ \bar{x} &= \frac{1}{(9/2)} \int_{-2}^1 x(-x^2 - (x-2)) dx = \frac{2}{9} \int_{-2}^1 (-x^3 - x^2 + 2x) dx \\ &= \frac{2}{9} \left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2 \right) \Big|_{-2}^1 \quad \textcircled{1} \\ &= \frac{2}{9} \left(\left(-\frac{1}{4} - \frac{1}{3} + 1 \right) - \left(-\frac{(-2)^4}{4} - \frac{(-2)^3}{3} + (-2)^2 \right) \right) \quad \textcircled{1} \end{aligned}$$



(b) [4 pts] Find the y -coordinate \bar{y} of the centroid of the region R . You do not need to simplify your answer.

$$\begin{aligned} \bar{y} &= \left(\frac{1}{(9/2)} \right) \int_{-2}^1 \frac{1}{2} \left\{ (-x^2)^2 - (x-2)^2 \right\} dx \quad \textcircled{1} \\ &= \frac{1}{9} \int_{-2}^1 (x^4 - (x^2 - 4x + 4)) dx = \frac{1}{9} \int_{-2}^1 (x^4 - x^2 + 4x - 4) dx \quad \textcircled{1} \\ &= \frac{1}{9} \left(\frac{x^5}{5} - \frac{x^3}{3} + 2x^2 - 4x \right) \Big|_{-2}^1 \quad \textcircled{1} \\ &= \frac{1}{9} \left\{ \left(\frac{1}{5} - \frac{1}{3} + 2 - 4 \right) - \left(\frac{(-2)^5}{5} - \frac{(-2)^3}{3} + 2(-2)^2 - 4(-2) \right) \right\} \quad \textcircled{1} \end{aligned}$$

6. The graph of a function $y = f(x)$ is the curve C that passes through the point $(0, -5)$ and whose slope at (x, y) is $\frac{x}{2ye^{y^2}}$.

(a) [6 pts] Find an equation of the curve C .

$$\frac{dy}{dx} = \frac{x}{2ye^{y^2}} \Rightarrow 2ye^{y^2} dy = x dx \quad \textcircled{1}$$

$$\int 2ye^{y^2} dy = \int x dx \Rightarrow e^{y^2} = \frac{x^2}{2} + C \quad \textcircled{1}$$

Since $(0, -5)$ satisfies the equation above, we

have $e^{(-5)^2} = \frac{0^2}{2} + C \Rightarrow C = e^{25} \quad \textcircled{1}$

Hence, an equation of the curve C is given

$$e^{y^2} = \frac{x^2}{2} + e^{25} \quad \textcircled{1} \quad (1)$$

(b) [2 pts] Find $f(x)$ by solving the equation in the part (a) above.

By (1), $y^2 = \ln\left(\frac{x^2}{2} + e^{25}\right) \quad \textcircled{1}$

since $y(0) < 0$, $y = -\sqrt{\ln\left(\frac{x^2}{2} + e^{25}\right)} = f(x) \quad \textcircled{1}$

7. [8 pts] Let $f(x)$ be a continuous function defined on $[3, 5]$. It is known that $0 \leq f'(x) \leq 2$ and $0 \leq f''(x) \leq 4$ for all x in $[3, 5]$. Let $g(x) = xf(x)$. How large should we take n in order to guarantee that the Trapezoidal Rule approximation for $\int_3^5 g(x) dx$ is accurate to within 0.25? You may use the fact: Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T is the error in the Trapezoidal Rule, for $\int_a^b f(x) dx$, then $|E_T| \leq \frac{K(b-a)^3}{12n^2}$.

Let $g(x) = xf(x) = xf$. Then

$$g' = f + xf', \quad g'' = f' + f' + xf'' = 2f' + xf'' \quad \textcircled{1}$$

$$|g''(x)| = 2f' + xf'' \leq 2 \cdot 2 + 5 \cdot 4 = 24 (=K) \quad \textcircled{2}$$

Hence, it is enough to find n such that

$$\frac{24 \cdot (5-3)^3}{12n^2} \leq 0.25 = \frac{1}{4} \quad \textcircled{1}$$

$$\text{or } \frac{2 \cdot 2^3}{n^2} \leq \frac{1}{4} \Rightarrow n^2 > 2^4 \cdot 2^2 \quad \textcircled{1}$$

$$\Rightarrow n \geq \sqrt{2^4 \cdot 2^2} = 2^2 \cdot 2 = 8 \quad \textcircled{1}$$

Thus, we can take $n = 8$ 8 \textcircled{1}