Midterm Exam #1—Math 101, Section SECTIONNUMBER
January 30, 2015
Duration: 50 minutes

Solution

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		Total	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Evaluate $\int (\sin x + \frac{1}{\sqrt{x}})dx$.

$$\int (\sin x + \frac{1}{\sqrt{x}})dx = \int \sin x\,dx + \int x^{-1/2}\,dx = -\cos x + 2x^{1/2} + C$$

1b. **[3 pts]** Evaluate $\int_0^2 f(x)dx$, where $f(x) = \begin{cases} x^5 & \text{for } -1 \le x \le 1; \\ 2x-1 & \text{for } 1 \le x \le 3. \end{cases}$

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 x^5\,dx + \int_1^2 (2x-1)\,dx$$

$$= \frac{x^6}{6}\Big|_0^1 + (x^2 - x)\Big|_1^2 = \frac{1}{6} + (2^2 - 2) = \frac{13}{6}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Evaluate $\int xe^{3x^2+4}\,dx$.

$$\int xe^{3x^2+4}\,dX \underset{\substack{u=3X^2+4 \; ① \\ du=6Xdx}}{=} \int e^u\left(\frac{du}{6}\right) = \frac{1}{6}e^u + C = \frac{1}{6}e^{3X^2+4} + C \quad ①$$

①

2b. **[3 pts]** Evaluate $\int_0^{\sqrt[3]{e-1}} \frac{x^2}{x^3+1}\,dx$.

$$\int_0^{\sqrt[3]{e-1}} \frac{X^2}{X^3+1}\,dX \underset{\substack{\text{let } u=X^3+1 \; ① \\ du=3X^2dX}}{=} \int_1^e \frac{du/3}{u} \; ① = \frac{1}{3}\ln|u|\Big|_1^e = \frac{1}{3} \; ①$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** A force of 30 N is required to hold a spring that has been stretched from its natural length 0.12 meters to a length 0.18 meters. How much is done in stretching the spring from 0.18 meters to 0.2 meters?

The force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$, where k is a constant. Since $0.18 - 0.12 = 0.06$ m, we have
$30 = f(0.06) = 0.06k \Rightarrow k = \frac{30}{0.06} = 500$. ①
Hence, $f(x) = 500x$. The work done in stretching the spring from 0.18 meters to 0.2 meters is

$$W = \int_{0.06}^{0.08} 500x \, dx ① = 500\left(\frac{x^2}{2}\right)\Big|_{0.06}^{0.08} = 250(0.08^2 - 0.06^2). ①$$

3b. **[3 pts]** A heavy rope, 70 ft long, weighs 0.8 lb/ft and hangs over the edge of a building 100 ft high. How much work is done in pulling the rope to the top of the building?

The portion of the rope from x ft to $(x+\Delta x)$ ft below the top of the building weigh $0.8\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $(0.8\Delta x)x_i^*$ ft-lb

The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (0.8x_i^*)\Delta x = \int_0^{70} 0.8x\,dx ① = 0.4x^2\Big|_0^{70} ①$$

$$= 0.4(70^2) \text{ ft-lb} ①$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Express $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{2i}{n}+\frac{i^3}{n^3}}$ as a definite integral. Do not evaluate it.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{2i}{n}+\frac{i^3}{n^3}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{2\left(\frac{i}{n}\right)+\left(\frac{i}{n}\right)^3}$$

$$= \int_0^1 \sqrt{2x+x^3}\, dx.$$

1/2

1/2

2

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. Let R be the region enclosed by the line $y = 2x$ and the parabola $8 - y^2 = 4x$.

5a. **[2 pts]** Find the intersection points of the line $y = 2x$ and the parabola $8 - y^2 = 4x$.

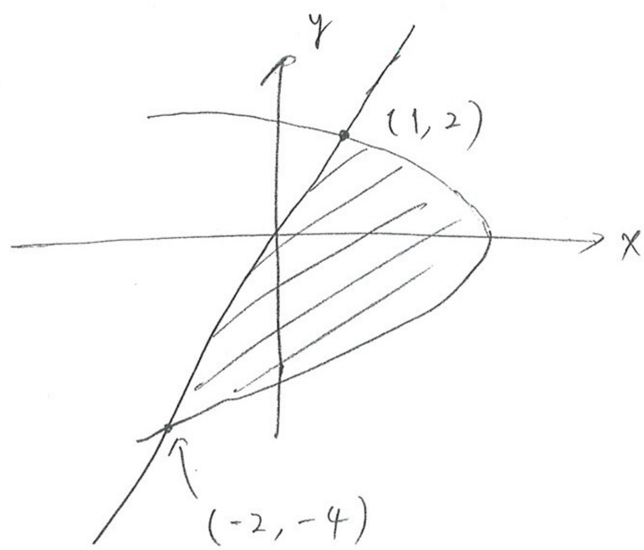
$$8-(2x)^2 = 4x \Rightarrow 8 - 4x^2 = 4x \Rightarrow x^2+x-2=0$$ 1

$$\Rightarrow x = 1 \text{ or } x = -2.$$

Hence, $(1,2)$ and $(-2,-4)$ are the intersection points.

1

5b. **[6 pts]** Find the area of the region R.



The area of the region R is

$$\int_{-4}^{2} \left[\left(2-\frac{1}{4}y^2\right)-\frac{1}{2}y\right]dy$$ 1 1 2

$$= \left(2y - \frac{1}{4}\cdot\frac{1}{3}y^3 - \frac{1}{2}\cdot\frac{1}{2}y^2\right)\Big|_{-4}^{2}$$ 1

$$= \left(2\cdot 2 - \frac{1}{12}(2^3) - \frac{1}{4}(2^2)\right) - \left(2(-4) - \frac{1}{12}(-4)^3 - \frac{1}{4}(-4)^2\right)$$

$$= \left(4 - \frac{2}{3} - 1\right) - \left(-8 + \frac{16}{3} - 4\right) = 9.$$ 1

6. Let R be the region enclosed by the curves $y = (\sqrt{3})\,x^2$ and $81x = y^2$.

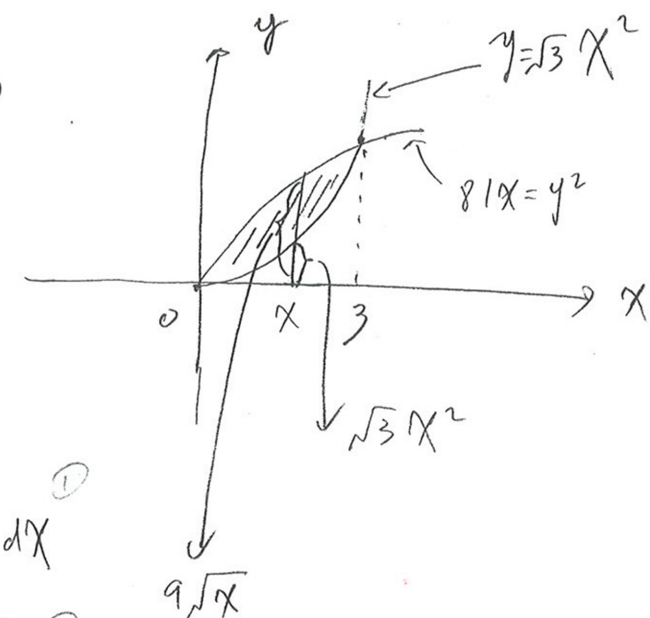
6a. **[2 pts]** Find the x-coordinates of the intersection points of the curves $y = (\sqrt{3})\,x^2$ and $81x = y^2$.

$$\underbrace{81x = (\sqrt{3}x^2)^2}_{①} = 3x^4 \Rightarrow \underbrace{x = 0 \text{ or } x = 3}_{①}$$

6b. **[6 pts]** Find the volume of the solid obtained by rotating the region R about the x-axis.

The cross-sectional area is

$$A(x) = \pi(9\sqrt{x})^2 - \pi(\sqrt{3}x^2)^2$$
$$= 81\pi x - 3\pi x^4$$



Therefore, the volume is

$$V = \int_0^3 A(x)\,dx = \int_0^3 (81\pi x - 3\pi x^4)\,dx$$
$$= \left(81\pi \frac{x^2}{2} - 3\pi \cdot \frac{x^5}{5}\right)\Big|_0^3 = \pi \cdot \frac{3^7}{10}.$$

7. **[8 pts]**

7a. **[4 pts]** Assume that $\frac{df}{dx} = f'(x)$ is continuous function, $\int_0^1 f'(3x+1)\,dx = 2$, and $f(1) = 3$. Evaluate $f(4)$.

$$\int_0^1 f'(3x+1)\,dx \underset{\substack{u=3x+1 \; ① \\ du=3\,dx}}{=} \int_1^4 f'(u)\,\frac{du}{3} \; ①$$

$$= \frac{1}{3} f(u)\Big|_1^4 \; ① = \frac{1}{3}\left(f(4)-f(1)\right) = \frac{1}{3}\left(f(4)-3\right)$$

Thus, $\frac{1}{3}(f(4)-3) = 2 \Rightarrow f(4)-3=6$ or $f(4)=9$. ①

7b. **[4 pts]** For what value of x does the graph of $y = g(x) = \int_{3x^2+2x}^{1} \frac{1}{t^2}\,dt$ have a horizontal tangent line?

(b) $$g'(x) = -\frac{d}{dx}\int_1^{3x^2+2x} \frac{1}{t^2}\,dt \; ①$$

$$= -\frac{1}{(3x^2+2x)^2}\cdot(6x+2) \; ①$$

$g'(x) = 0$ ① $\Rightarrow 6x+2=0 \Rightarrow x = -\frac{1}{3}$. ①