The University of British Columbia

Midterm Examination (Version 1) - November 8, 2012

Closed book examination | Time: 60 minutes

Last Name ______________________ **First** ____________________

Student Number ______________________ **Signature** ____________________

MATH 104 or MATH 184 (Circle one) **Section Number:** __________

Special Instructions:

No memory aids, calculators, or electronic devices of any kind are allowed on the test. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. Numerical answers should be left in calculator-ready form, unless otherwise indicated. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1		9
2		~~10~~ 9
3		~~8~~ 10
4		~~9~~ 8
5		14
Total		50

1. (a) (5 points) Consider the function

$$f(x) = e^{(x-1)^2}$$

Find the critical points of $f(x)$, and determine the absolute maximum and minimum values of $f(x)$ on the interval $[-1, 2]$.

$$f' = e^{(x-1)^2} \cdot 2(x-1) = 0$$

$$\Rightarrow x = 1$$

(f' always defined)

$\therefore$ critical pts are $x = 1$

$$\begin{cases} f(-1) = e^4 \\ f(1) = e^0 \\ f(2) = e^1 \end{cases}$$

$\therefore$ by E.V.T. absolute max is $f(-1) = e^4$

absolute min is $f(1) = e^0$

(b) (4 points) Consider the curve

$$2x^3y + xy^2 = 8$$

Determine the equation of the tangent line to the curve at the point $(1, 2)$.

$$6x^2y + 2x^3y' + y^2 + x2yy' = 0$$

$$\Rightarrow \quad (2x^3 + 2xy)y' = -6x^2y - y^2$$

$$\Rightarrow \quad y' = \frac{-6x^2y - y^2}{2x^3 + 2xy}$$

$$\Rightarrow \quad y'(1) = \frac{-6\cdot 2 - 2^2}{2 + 2\cdot 2} = \frac{-16}{6} = \frac{-8}{3}$$

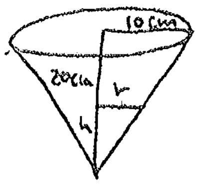
$\therefore$ tangent line is

$$\frac{y-2}{x-1} = -\frac{8}{3}$$

$$\boxed{y - 2 = \frac{-8}{3}(x-1)}$$

2. Beer flows out the bottom of a funnel (a right circular cone with tip pointing downwards) with radius 10 cm and height 20 cm. You measure the volume and depth of the beer in the funnel as it flows. Let V be the volume of beer in the funnel and h its depth at time t.

   (a) (3 points) Express V as a function of h, then use this to express $\frac{dV}{dt}$ in terms of h and $\frac{dh}{dt}$.



$$V = \frac{1}{3}\pi r^2 h$$

By similar triangles, $\frac{r}{h} = \frac{10\text{ cm}}{20\text{cm}}$

$$\Rightarrow r = \frac{1}{2}h.$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3.$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

   (b) (2 points) For some constant C it is known that the beer flow satisfies

$$\frac{dV}{dt} = C\sqrt{h}. \qquad (Torricelli's Law)$$

   Is C positive or negative (explain)? If h is measured in centimeters and t is measured in seconds, what are the units of C?

- C is negative because V is decreasing.
- Units of C are $\frac{cm^3}{s} \cdot \frac{1}{\sqrt{cm}} = \boxed{\frac{cm^{5/2}}{s}}$.

(c) (3 points) Show that

$$\frac{dh}{dt} = \frac{4C}{\pi} h^{-3/2}.$$

From (a) and (b),

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} = C\sqrt{h}.$$

$$\therefore \frac{dh}{dt} = \underbrace{\frac{4C}{\pi} \frac{\sqrt{h}}{h^2}}_{\text{full points.}} = \frac{4C}{\pi} h^{-3/2}.$$

(d) (1 point) In an experiment to determine the value of C, you measure the beer level in the funnel to be falling at a rate of 5 cm/s when $h = 4$ cm. Determine the value of C.

From c,

$$C = \frac{\pi}{4} h^{3/2} \frac{dh}{dt}.$$

$$= \frac{\pi}{4} (4\,cm)^{3/2} \cdot (-5\,cm/s).$$

$$= -\frac{\pi}{4}\, 8\, cm^{3/2} \cdot 5\, cm/s.$$

$$= -10\pi\, \underline{cm^{5/2}/s}.$$

3. The price p (in dollars) for a bag of cat treats and the hourly number of sales q are related by $q = \sqrt{36 - 9p}$. Currently, you charge \$3 per bag.

(a) (3 points) Compute the elasticity of demand ϵ for the cat treats at this price.

$$\varepsilon = \frac{p}{q}\, q'(p)$$

$$q' = \frac{1}{2\sqrt{36-9p}}(-9)$$

$$p=3 \Rightarrow q = 3, \quad q' = \frac{1}{6}(-9) = -\frac{3}{2}$$

$$\therefore \quad \varepsilon(3) = \frac{3}{3}\left(-\frac{3}{2}\right) = -\frac{3}{2}$$

(b) (2 points) Suppose the price is raised from \$3 and this yields a 3% decrease in demand. Estimate the percentage by which the price was changed.

% change in q $= \left(-\frac{3}{2}\right)$ % change in p

$\therefore \quad -3\% = \left(-\frac{3}{2}\right)$ % change in p

$\therefore$ % change in p $= 2\%$

(c) Now suppose $\epsilon = -1/104$ at a certain price for the cat treats. For part (iv) *include an explanation or no credit will be given.*

i. (1 point) State whether the cat treats are price elastic, price inelastic or price unit elastic at this price.

Answer: price inelastic

ii. (1 point) If this price is raised slightly, state whether the revenue will increase, decrease, stay constant, or whether there is insufficient information to detemine this.

Answer: revenue increases

iii. (1 point) If this price is raised slightly, state whether the demand will increase, decrease, stay constant, or whether there is insufficient information to detemine this.

Answer: demand decreases

iv. (2 points) If this price is raised slightly, state whether the profit will increase, decrease, stay constant, or whether there is insufficient information to detemine this. Assume here that the cost function $C(q)$ is increasing.

Answer: profit increases

provide an explanation in the space below

$P = R - C$

At given price,

$p\uparrow \Rightarrow R\uparrow$ by ii

also $p\uparrow \Rightarrow q\downarrow$

$\Rightarrow C\downarrow$ by Assumption

$\therefore p\uparrow \Rightarrow \underline{P} = R - C \uparrow$

4. In the year 2000, a student takes out a \$20 000 loan at an annual interest rate of r (interest compounded continuously). The student makes one loan payment of \$40 000 in 2015, then one more loan payment of \$$L$ in 2020 after which the student owes nothing.

(a) (3 points) If the student's debt is \$40 000 in 2005, what is r?

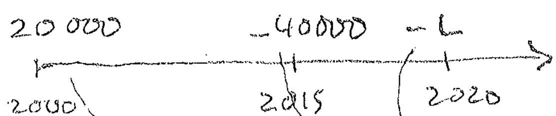
$$20\,000\, e^{r(5)} = 40\,000$$
$$\Rightarrow e^{r(5)} = 2$$
$$\Rightarrow r(5) = \ln 2$$
$$\boxed{r = \ln 2/5}$$

(b) (2 points) At what rate (in dollars per year) is the student's debt increasing in 2010? (use the value of r from part (a))

let $A(t) = 20\,000\, e^{\ln 2/5\, t}$

$$\Rightarrow A'(t) = 20\,000\,(\ln 2/5)\, e^{\ln 2/5\, t}$$
$$\Rightarrow \boxed{A'(10) = 20\,000\,(\ln 2/5)\, e^{\ln 2/5\,(10)} = 16\,000 \ln 2 \quad \$/yr}$$

(c) (3 points) What is L? (use the value of r from part (a))



$$-L$$
$$-40000\, e^{r(5)}$$
$$+20000\, e^{r(20)}$$
$$= 0$$

$$\therefore -L - 40000\, e^{r(5)} + 20\,000\, e^{r(20)} = 0$$

$$\boxed{L = -40000\, e^{r(5)} + 20000\, e^{r(20)} \ \$}$$

(Substitute $r = \ln 2/5$ above)

or else,

$\Rightarrow$ Amt. owing after 1st payment
$$= \left(20\,000\, e^{r(15)} - 40000\right)$$

$\Rightarrow \therefore$ Amt. owing after 2nd payment
$$= \left(20\,000\, e^{r(15)} - 40000\right) e^{r(5)} - L$$
$$= 0$$

$$\therefore \boxed{L = \left(20\,000\, e^{r(15)} - 40\,000\right) e^{r(5)} \ \$}$$

5. Consider the following function

$$f(x) = \frac{x^2 - 4}{x^2 - 9}$$

You may assume that

$$f'(x) = \frac{-10x}{(x^2-9)^2}; \qquad f''(x) = \frac{30(x^2+3)}{(x^2-9)^3}.$$

(a) (1 point) Find the x- and y-intercepts of the graph of the function.

$y_{int} = \frac{4}{9}$

$x_{int} = \pm 2$

(b) (1 point) Find the domain of $f(x)$

all $x \neq \pm 3$

(c) (3 points) Find all critical points, and intervals of increase and decrease for $f(x)$.

CP $x = 0$

$x = \pm 3$

$(-\infty, -3)$ incr.

$(-3, 0)$ incr.

$(0, +3)$ decr.

$(+3, \infty)$ decr.

(d) (3 points) Find the intervals of concavity for $f(x)$

$(-\infty, -3)$ up

$(-3, 3)$ down

$(3, \infty)$ up

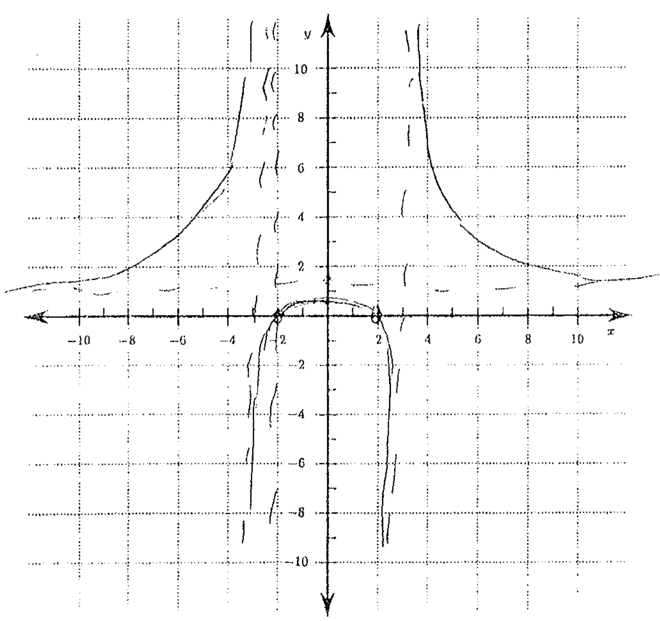
(e) (2 points) Find any horizontal or vertical asymptotes

v.a. $x = \pm 3$ since $\lim_{x \to \pm 2} |f(x)| = \infty$

h.a. $y = 1$; since

$\begin{cases} \lim_{x \to \infty} (f(x)) = 1 \\ \lim_{x \to -\infty} (f(x)) = 1 \end{cases}$

(f) (4 points) Sketch $y = f(x)$ carefully and neatly in the grid provided. Your sketch should indicate where $f(x)$ is positive and negative, and also each of the features determined in parts (a)-(e) above.



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