Solutions to Math 184 Midterm 2 (Version 2)

(Nov. 14, 2019)

(i) \( \frac{d}{dx} (\cos y + 4x) = \frac{d}{dx} (y^2) = y \cos y + 4 = 2y y' \). At the point \( \left( \frac{\pi}{5}, \pi \right) \), we have \( y \cos \pi + 4 = 2 \pi y' \Rightarrow -y' + 4 = 2 \pi y' \Rightarrow 2 \pi y' + y' = 4 \Rightarrow y' = \frac{4}{1 + 2 \pi} \). Answer: A

(ii) \( 3 \cos t = 10000 e^{0.04t} \Rightarrow \frac{7}{2} = \frac{3500}{1000} = e^{0.04t} \Rightarrow \ln \left( \frac{7}{2} \right) = 0.04t \Rightarrow t = \frac{\ln 3.5}{0.04} = \frac{1000}{64} \ln \frac{7}{2} = 25 \ln \frac{7}{2} \). Answer: D

(iii) \( f' = -\frac{2}{x^3} + \frac{1}{x} = -\frac{2 + x}{x^3} \). \( f' = 0 \Rightarrow x = \pm \sqrt{2} \). Since \( \sqrt{2} \) is not in the domain of \( f \), \( \sqrt{2} \) is the unique critical point of \( f \). Answer: A

(iv) \( f'(2) = 0 \) and \( g'(x) = 6x + 6 + f'(x) = 2g'(2) = 6 \cdot 2 + 6 + f'(2) = 18 \). Answer: D

(v) Since \( x \to 2 \), \( x \to 4 \to -1 \) and \( x \to 3^+ \) as \( x \to 3^+ \), the limit is \( -\infty \). Answer: A
(vi) \[ \lim_{x \to 0} f(x) = \lim_{x \to \infty} \frac{6x^2 - 9}{x(2x - 3)} = \lim_{x \to \infty} \frac{6}{2} - \frac{9}{x} = \frac{6}{2} = 3 \]

\[ (= \lim_{x \to \infty} f(x)) \text{. Hence, } y = 3 \text{ is the equation for the horizontal asymptote of } f(x). \]

\[ \text{Answer: B} \]

(vii) \[ f' = 3x^2 + 2x, \quad f'' = 6x + 2 \]

Since \( f'' > 0 \) if and only if \( x > -\frac{1}{3}, \)

\((-\frac{1}{3}, \infty) \) is the interval on which \( f(x) \)

is concave up.

\[ \text{Answer: B} \]
2. \[
\frac{d}{dt} \left( \frac{1}{R} \right) = \frac{d}{dt} \left( \frac{1}{R_1} \right) + \frac{d}{dt} \left( \frac{1}{R_2} \right) \\
- \frac{1}{R_1^2} \frac{dR_1}{dt} = - \frac{1}{R_1^2} \frac{dR_1}{dt} - \frac{1}{R_2^2} \frac{dR_2}{dt} \\
\text{or } \frac{1}{R^2} \frac{dR}{dt} = \frac{1}{R_1^2} \frac{dR_1}{dt} + \frac{1}{R_2^2} \frac{dR_2}{dt} = \frac{20}{R_1^2} + \frac{40}{R_2^2} \tag{1}
\]

When \( R_1 = 1 \text{m} \) and \( R_2 = 4 \text{m} \), we have
\[
R = \sqrt{\left( \frac{1}{R_1} \right) + \left( \frac{1}{R_2} \right)} = \frac{1}{\sqrt{(\frac{1}{1}) + (\frac{1}{4})}} = \frac{4}{\sqrt{17}} = 0.80. \tag{2}
\]

By (1) and (2), we get
\[
\frac{dR}{dt} = R^2 \left( \frac{20}{R_1^2} + \frac{40}{R_2^2} \right) = 0.8^2 \left( \frac{20}{1^2} + \frac{40}{4^2} \right) \left( \text{s}^{-1} \right).
\]
3. \[ y' = \frac{2x+1}{x^2+x+1} \]. Noting that the denominator \( x^2+x+1 \) is always positive, we get the only critical number \( x = -\frac{1}{2} \), at which \( y' = 0 \).

Checking the \( y \) values at the two endpoints of the interval and the critical point:

\[ y(-1) = \ln(1-1+1) = \ln 1 = 0, \]
\[ y\left(-\frac{1}{2}\right) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln \frac{3}{4} < 0, \]
\[ y(1) = \ln(1^2+1+1) = \ln 3 > 0, \]

we see that \( \ln \frac{3}{4} \) is the absolute minimum and \( \ln 3 \) is the absolute maximum.
Solution: Implicit differentiation of the demand curve with respect to $p$:

\[ \frac{q'}{q} - \frac{1}{p} + 0.01 = 0, \]

which leads to $q' = q(1/p - 0.01)$. The elasticity is $\varepsilon = \frac{p}{q} \cdot q' = p(1/p - 0.01) = 1 - 0.01p$. At the current price of $p = 120$, $\varepsilon = 1 - 0.01 \times 120 = -0.2$. The product is price inelastic.

(ii) [2]

Solution: The elasticity

\[ \varepsilon = \frac{q}{p} \cdot \frac{dq}{dp} = \frac{dq/q}{dp/p} \approx \frac{\Delta q/q}{\Delta p/p} \]

can be interpreted as the ratio between the relative changes of demand to price. Thus, when the price lowers by 5% (i.e. $\Delta p/p = -0.05$), the percentage change in demand is

\[ \Delta q/q \approx \varepsilon \cdot \Delta p/p = (-0.2) \times (-0.05) = 0.01. \]

The demand will increase by 1%.

(iii) [1]

Solution: Since the product is price-inelastic at the current pricing, we should raise the price to increase the revenue.

(iv) [2]

Solution: The maximum revenue is achieved at $\varepsilon = 1 - 0.01p = -1$. The price is $p = 200$. Alternatively, the student may write out $R = pq$ and set $MR(p) = q + p q' = 0$. At the start, the student could also have solved for $q$ in terms of $p$: $q = pe^{(0.01)p}$, and then avoid the implicit differentiation in answering (a) and onward.

Alternative demand curves if we wish to tweak the problem: $q = 100 - 2\sqrt{p}$ at $p = 100$; $qp + 30p + 50q = 8500$ at $p = 150$; $q = 1000e^{-p/200}$, at $p = 100$. 
5. (i) \( x = 2 \) is the vertical asymptote.

(ii) \( f(x) \) does not have a horizontal asymptote.

(iii) \( f'(x) \) does not have a horizontal asymptote.

\[
f'(x) = x^2 - 4 \Rightarrow x = 0 \text{ and } x = 4 \text{ are critical points of } f.
\]

\[
\begin{array}{cccccc}
& + & x & - & x & + \\
f' & 9 & 0 & 2 & 4 & 9
\end{array}
\]

Hence, \( f \) is decreasing on \((0, 2)\) and \((2, 4)\), and \( f \) is increasing on \((-\infty, 0)\) and \((4, \infty)\).

(iv) \( f'' = \frac{(x^2 - 4x)'(x^2 - 4x) - (x^2 - 4x)'(x^2 - 4x)'}{(x-2)^4} \)

\[
= \frac{(2x-4)(x^2 - 4x) - (x^2 - 4x)^2(2)(x-2)}{(x-2)^4} = \frac{8}{(x-2)^3}
\]

\( f \) is concave up on \((2, \infty)\), \( f \) is concave down on \((-\infty, 2)\).

(v) \( f'' \) is undefined at \( x = 2 \) and \( x = 4 \).
6. \( \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \implies 2x + 2yy' = 0 \implies y' = -\frac{x}{y}. \)

So the tangent line at \((a, b)\) has the slope \(-\frac{a}{b}\), and the tangent line at \((a, -b)\) has the slope \(-\frac{a}{b} = \frac{a}{b}\).

The two tangent lines are perpendicular when \(\frac{-a}{b} \cdot \frac{a}{b} = -1\) or \(a^2 = b^2\).

Plugging that back into the equation \(x^2 + y^2 = 1\), we get \(2a^2 = 1\) so \(a = \pm \frac{1}{\sqrt{2}}\). Likewise for \(b\), we have the four points:

\((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) and \((-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\) (or \((\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})\)).