The University of British Columbia
March 16, 2017
Common Midterm for All Sections of MATH 105 (Version 1)

Closed book examination

Last Name __________________________ First __________________

Signature __________________________

Student Number ______________________

MATH 105  Section Number:___________

Special Instructions:
No memory aids are allowed. No calculators. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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12. Short-Answer Questions: Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) [3] Fill in the blanks with left, right or midpoint, an interval, and a value of \( n \) in the statement below:

\[
\sum_{k=1}^{4} f(3+k) \cdot 1 \text{ is a } \underline{\text{________}} \text{ Riemann sum for } f(x) \text{ on the interval } [\underline{\text{____}}, \underline{\text{____}}] \text{ with } n = \underline{\text{________}}.
\]

(b) [3] Evaluate \( \int_{0}^{12} f(t) \, dt \), where the graph of \( f \) is given in the following figure:

![Graph of f(t)](image)

Answer:
(c) [3] Evaluate $\int \frac{x^2}{\sqrt{1 - 2x^3}} \, dx$.

Answer:

(d) [3] Evaluate $\int \frac{\ln x}{x^9} \, dx$.

Answer:
2. Suppose that \( \int_{-3}^{-2} f(x) \, dx = -2, \int_{-2}^{0} f(x) \, dx = 4, \) \( f(x) \leq 0 \) on the interval \([-3, -2]\) and \( f(x) \geq 0 \) on the interval \([-2, 0]\).

(a) Evaluate \( \int_{-3}^{0} f(x) \, dx \).

(b) Evaluate \( \int_{-3}^{0} |f(x)| \, dx \).

(c) Evaluate \( \lim_{n \to \infty} \sum_{k=1}^{n} 3f \left( -2 + \left( k - 1 \right) \frac{2}{n} \right) \frac{2}{n} \).

(a) Evaluate $\int \sin^3 x \cos^{2016} x \, dx$.

(b) Evaluate $\int \sec^2 x e^{\tan x} \, dx$. 

(c) Approximate \( \int_{1}^{3} e^{\frac{1}{x}} \, dx \) using the Trapezoid rule with 4 subintervals.

(d) Evaluate \( \int \frac{13x - 12}{x^2 - 3x + 2} \, dx \).
[9] 4. Evaluate the following integrals.

(a) \( \int_{8}^{\infty} x^{-\frac{5}{3}} \, dx \).

(b) \( \int_{1}^{e} (\ln x)^2 \, dx \).
(c) $\int \sqrt{9 - x^2} \, dx$. 
[6] 5. Let $y = y(t)$ be a function defined on $\left[\frac{1}{2}, \infty\right)$. If $\frac{dy}{dt} = \frac{2t^2 + 4}{t} y$ and $y(1) = 1$, find the function $y = y(t)$. 

(a) Suppose that \( f(x) \) and \( g(x) \) are continuous functions on \(( -\infty, \infty )\) such that

\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx \quad \text{for all real numbers } a \text{ and } b.
\]

Is it possible that \( f(x) \) and \( g(x) \) are different functions? Please justify your answer.

(b) Let \( F(x) = \int_0^x te^{-t^2} \, dt \) for \( x \) in \(( -\infty, \infty )\). Find the second derivative \( F''(x) \) of \( F(x) \).
1 - \sin^2 x = \cos^2 x, \quad 1 + \tan^2 x = \sec^2 x, \quad \sec^2 x - 1 = \tan^2 x

Half-angle formulas:
\[
\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}
\]

Double-angle formulas:
\[
\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x
\]

Indefinite integrals:
\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C, \quad \int \csc x \, dx = -\ln |\csc x + \cot x| + C
\]

Summation identities:
\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4} = \left( \frac{n(n + 1)}{2} \right)^2.
\]

Approximation’s Rules and their error bound: Let \( y = f(x) \) be a continuous function defined on the interval \([a, b]\) such that its derivatives \( f'' \) and \( f^{(4)} \) are continuous. The midpoint Rule approximation \( M(n) \), the Trapezoid Rule approximation \( T(n) \) and the Simpson’s Rule approximation \( S(n) \) to \( \int_{a}^{b} f(x) \, dx \) using \( n \) equally spaced subintervals on \([a, b]\) are given by the following formulas:
\[
M(n) = \sum_{k=1}^{n} f \left( a + (k - \frac{1}{2}) \Delta x \right) \Delta x,
\]
\[
T(n) = \left( \frac{1}{2} f(x_0) + \sum_{k=1}^{n-1} f(a + k \Delta x) + \frac{1}{2} f(x_n) \right) \Delta x,
\]
\[
S(n) = \left( f(x_0) + 4 f(x_1) + 2 f(x_2) + 4 f(x_3) + ... + 4 f(x_{n-1}) + f(x_n) \right) \Delta x \frac{3}{3},
\]
where \( \Delta x = \frac{b - a}{n} \). The absolute errors in approximating the integral \( \int_{a}^{b} f(x) \, dx \) by the Midpoint Rule, Trapezoid Rule satisfy the inequalities
\[
E_M \leq \frac{k(b - a)}{24} (\Delta x)^2 \quad \text{and} \quad E_T \leq \frac{k(b - a)}{12} (\Delta x)^2,
\]
where \( k \) is a real number such that \( |f''(x)| \leq k \) for all \( x \) in \([a, b]\).
The absolute errors in approximating the integral $\int_{a}^{b} f(x)dx$ by the Simpson’s Rule satisfies the inequalities

$$E_S \leq \frac{K(b-a)}{180} (\Delta x)^4$$

where $K$ is a real number such that $|f^{(4)}(x)| \leq K$ for all $x$ in $[a, b]$.

Equivalently, the error bounds can also be written as:

$$E_M \leq \frac{k(b-a)^3}{24n^2},$$

$$E_T \leq \frac{k(b-a)^3}{12n^2},$$

$$E_S \leq \frac{K(b-a)^5}{180n^4}.$$

(you can use any of these two versions for error bounds, as they give the same result.)
Solutions to Math 105 Midterm 2 (Version I)

(a) \[ \sum_{k=1}^{4} f(3+k) \cdot 1 \text{ is a right Riemann sum for } f(x) \text{ on the interval } [3, 7] \text{ with } n = 4 \]

(b) \[ -\frac{1}{4} \pi \cdot 2^2 + \frac{1}{2} \cdot 6 \cdot 3 - \frac{1}{2} \cdot 4 \cdot 2 = -\pi + 9 - 4 = 5 - \pi \]

(c) \[ \int_{\frac{1}{\sqrt{1-2x^2}}}^{\frac{1}{\sqrt{1-2x^2}}} u = 1-2x^2 \, du = -6x^2 \, dx \]

\[ = -\frac{1}{6} \cdot \frac{u^{3/2}}{1/2} + C = -\frac{1}{3} \sqrt{1-2x^2} + C \]

(d) \[ \int \frac{\ln x}{x^9} \, dx = -\frac{\ln x}{8x^8} - \int \frac{1}{x} \left( -\frac{1}{8} x^{-8} \right) \, dx \]

\[ = -\frac{\ln x}{8x^8} + \frac{1}{8} \int x^{-9} \, dx = -\frac{\ln x}{8x^8} - \frac{1}{64} \frac{1}{x^8} + C \]
(a) \[
\int_{-3}^{0} f(x) \, dx = \int_{-3}^{-2} f(x) \, dx + \int_{-2}^{0} f(x) \, dx = 2 + 4 = 2
\]

(b) \[
\int_{-3}^{0} |f(x)| \, dx = \int_{-3}^{-2} |f(x)| \, dx + \int_{-2}^{0} |f(x)| \, dx
\]

(c) (i) Since \[
\sum_{k=1}^{n} 3 \left( -2 + (k-1) \frac{2}{n} \right) \frac{2}{n} \]

is the Riemann sum for \(3f(x)\) on \([-2, 0]\), we have

\[
\lim_{n \to \infty} \sum_{k=1}^{n} 3 \left( -2 + (k-1) \frac{2}{n} \right) \frac{2}{n} = \int_{-2}^{0} 3f(x) \, dx
\]

\[
= 3(4) = 12.
\]
\[ \begin{align*}
3. \text{(a)} & \quad \int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \sin x \cos^2 x \, dx \quad (u = \cos x) \\
& = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\
& = - \int (\cos^{2018} x - \cos^{2016} x) \, du = - \frac{\cos^{2017} x}{2017} + \frac{\cos^{2019} x}{2019} + C \\
& = \frac{\cos^{2017} x}{2017} + \frac{\cos^{2019} x}{2019} + C
\end{align*} \]

\[ \begin{align*}
(\text{b)} & \quad \int \sec^2 x \tan x \, dx = \int e^u \, du = e^u + C \\
& \quad u = \tan x \\
& \quad du = \sec^2 x \, dx
\end{align*} \]

\[ \begin{align*}
\text{(c)} & \quad \Delta x = \frac{b-a}{n} = \frac{3 - 1}{4} = \frac{1}{2}. \quad \text{Our grid points are} \\
& \quad 1, \frac{3}{2}, 2, \frac{5}{2}, 3. \quad \text{Using the formula, we get} \\
& \quad \int_1^3 e^{\frac{1}{4}x} \, dx \approx \frac{1}{2} (\frac{1}{2} e^{1/2} + e^{3/2} + e^{5/2} + \frac{1}{2} e^{3/2})
\end{align*} \]

\[ \begin{align*}
\text{(d)} & \quad \frac{13X-12}{(x-2)(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+1} \\
13X-12 & = A(x-2) + B(x-1) + C (x+1) \\
\text{At } x = 2 & \Rightarrow B = 14; \quad \text{At } x = 1 & \Rightarrow A = 1. \\
\int \frac{13X-12}{x^2 - 3x + 2} \, dx & = \int \frac{-1}{x-1} \, dx + \int \frac{14}{x-2} \, dx = -\ln|x-1| + 14\ln|x-2| + C
\end{align*} \]
4 \hspace{1cm} \int_{8}^{\infty} x^{-5/3} \, dx = \lim_{t \to \infty} \int_{8}^{t} x^{-5/3} \, dx

= \lim_{t \to \infty} \left( -\frac{3}{2} x^{-2/3} \right) \bigg|_{8}^{t} = \lim_{t \to \infty} \left( -\frac{3}{2t^{2/3}} + \frac{3}{2 \cdot 8^{2/3}} \right)

= 0 + \frac{3}{2 \cdot 2^2} = \frac{3}{8}

(b) \hspace{1cm} \int_{1}^{e} (\ln x)^{2} \, dx = \left[ x(\ln x)^{2} \right]_{1}^{e} - 2 \int_{1}^{e} \ln x \, dx

u = (\ln x)^{2}, \quad v = 1

u' = 2(\ln x)^{1}, \quad v' = x

= e - 2 \left( (x \ln x) \bigg|_{1}^{e} - \int_{1}^{e} \ln x \, dx \right) = e - 2 \left( e - (e - 1) \right) = e - 2

(c) \hspace{1cm} \int_{0}^{\pi/2} \sqrt{9 - x^{2}} \, dx = \int_{0}^{\pi/2} \sqrt{9 - (3 \sin \theta)^{2}} \cdot 3 \cos \theta \, d\theta

\theta = 3 \sin \theta

= \int_{0}^{\pi/2} 3 \cos \theta \cdot 3 \cos \theta \, d\theta = 9 \int_{0}^{\pi/2} \cos^{2} \theta \, d\theta = 9 \int_{0}^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta

= \frac{9}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{9}{2} \left( \theta + \sin \theta \cos \theta \right) + C

= \frac{9}{2} \left( \frac{x^{2}}{2} + \frac{x}{3} \cdot \frac{\sqrt{9 - x^{2}}}{3} \right) + C.
\[ J \left[ \frac{dy}{y} \right] = \frac{2t^2 + 4}{t} \]  

\[ \int \frac{dy}{y} = \int \left( 2t^2 + \frac{4}{t} \right) dt \]

\[ \ln |y| = t^2 + 4 \ln t + C \]

\[ |y| = e^{t^2 + 4 \ln t + C} \]

\[ y = (\pm e^C) e^{t^2 + 4 \ln t} = k e^{t^2 + 4 \ln t}, \quad k = \pm e^C \]

\[ y(1) = k e^{1 + 4(0)} = ke^1 \Rightarrow k = \frac{1}{e} \]

\[ \text{Hence, } y = \frac{1}{e} e^{t^2 + 4 \ln t} = t^2 + e^{t^2 - 1} \]
6. (a) NO. In fact, for any \( t \) in \((\infty, \infty)\), we have

\[
\int_a^t f(x) \,dx = \int_a^t g(x) \,dx
\]

by the assumption. Hence, we get

\[
f(t) = \frac{d}{dt} \left( \int_a^t f(x) \,dx \right) = \frac{d}{dt} \left( \int_a^t g(x) \,dx \right) = g(t).
\]

(b) \( F'(x) = \left( x \int_0^x e^{-t^2} \,dt \right)' = \int_0^x e^{-t^2} \,dt + x e^{-x^2} \)

(49)

\[
F''(x) = e^{-x^2} + e^{-x^2} + x e^{-x^2} (2x) = (2 - 2x^2)e^{-x^2}.
\]