The University of British Columbia
Final Examination - December 8, 2017
Mathematics 184
All Sections

Closed book examination

Last Name ____________ First ________ Signature __________________

Student Number ______ Section Number ______ Instructor ____________

Special Instructions:
No books, notes, or calculators are allowed. A formula sheet is included.

Senate Policy: Conduct during examinations
- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1a, b, c, d, e</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>1f, g, h, i, j</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>1k, l, m, n</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. Short-Answer Questions. Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) If \( \lim_{x \to a^-} f(x) = L \), \( \lim_{x \to a^+} f(x) = 2L - 3 \) and \( \lim_{x \to a} f(x) \) exists, where \( L \) is a real number, find the value of \( L \).

Answer:

(b) Suppose

\[
f(x) = \begin{cases} 
\frac{1}{x} & \text{if } x \leq 2 \\
x + 5 & \text{if } x > 2
\end{cases}
\]

Let \( A = \lim_{x \to 2^-} f(x) \) and \( B = \lim_{x \to 2^+} f(x) \). Find \( A \) and \( B \).

Answer:
\[
A = \ , \ B =
\]

(c) Evaluate \( \lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} \).

Answer:
(d) Determine the limit $\lim_{x \to 4^-} \frac{(x - 2)(x - 5)}{x - 4}$.

Answer:

(e) Let

$$g(x) = \begin{cases} 
    2x^2 - 1 & \text{if } x < 2 \\
    a & \text{if } x = 2 \\
    x + 3 & \text{if } x > 2
\end{cases}$$

where $a$ is a constant. Is there a value of $a$ for which $g$ is continuous at 2?

Answer:

(f) Determine an equation of the tangent line to the graph of $f(x) = \frac{1}{5 - 2x}$ at the point $(2, 1)$.

Answer:
(g) Consider the following population function: \( p(t) = \frac{200t}{t + 2} \). What is the instantaneous growth rate at \( t = 5 \)?

Answer:

(h) Find \( \frac{dy}{dx} \), where \( y = e^{5x} \cos x \).

Answer:

(i) Find \( \frac{dy}{dx} \), where \( y = (e^x)^6 + e^{(x^6)} \).

Answer:
(j) Find \( \frac{dy}{dx} \) when \( \tan(xy) = 2x + y \).

(k) Two thousand dollars are deposited in a saving account at 7% annual interest rate compounded continuously. How many years are required for the balance in the account to reach $8000?

Answer:
(l) The edges of a cube increase at a rate of 3 cm/s. How fast is the volume changing when the length of each edge is 10 cm?

Answer:

(m) Find \( \frac{dy}{dx} \), where \( y = \ln(\tan^{-1} x) \). Note that another notation for the inverse tangent function \( \tan^{-1} x \) is \( \arctan x \).

Answer:

(n) Let \( A \) be the absolute maximum value of \( f(x) = x^7 e^{-x} \) on the interval \([-1, 5]\), and let \( B \) be the absolute minimum value of \( f(x) = x^7 e^{-x} \) on the interval \([-1, 5]\). Find \( A \) and \( B \).

Answer: 
\[ A = \ldots, B = \ldots \]
Full-Solution Problems. In questions 2 – 6, show your work. No credit will be given for the answer without the correct accompanying work.

[8] 2. Let \( f(x) = \frac{1}{5 - 3x} \). Use the definition of the derivative to find \( f'(2) \). No credit will be given for the use of any differentiation rule.
3. Let \( y = f(x) = (x - 3)(x + 3)^2 \).

(a) Does \( f(x) \) have any horizontal asymptote?

(b) Does \( f(x) \) have any vertical asymptote?

(c) Find the critical points of \( f(x) \).
(d) Find the interval or intervals on which \( f(x) \) is increasing.

(e) Find the interval or intervals on which \( f(x) \) is concave up.

(f) Sketch the graph of \( y = f(x) \) and indicate where all local maxima, local minima and inflection points occur.
4. Assume that the price $p$ and the demand $q$ of a certain good are related by the equation:

$$q = 10e^{1-p} + \frac{8}{p + 1}.$$ 

(a) Find the price elasticity $\mathcal{E}(p)$ as a function of $p$.

(b) If $p = 1$, does increasing price slightly increase revenue?

(c) Find the second order Taylor polynomial for the revenue function $R(p)$ centered at $p = 1$.

(d) If the demand is increasing at a rate of 24 units per week when the price is one dollar, find the rate of change of the price.
[10] 5. A carpenter has been asked to build a box with a square base. The sides of the box will cost $4 per square meter, the base of the box will cost $2 per square meter, and the top of the box will cost $3 per square meter. What are the dimensions of the box of maximal volume that can be constructed for $240?
6. Let $f(x)$ be a continuous function on $(-\infty, \infty)$. Suppose $\lim_{x\to 2017} (x + f(x)) = 2018$. Does the horizontal line $y = 2017$ intersect the graph of the function $y = (2018x^2 - x^3)f(x)$ at least twice? You need to justify your answer in detail to get credits.
(a) \[ L = 2L - 3 \Rightarrow L = 3. \]

(b) \[ A = \lim_{x \to 2} \frac{1}{x} = \frac{1}{2}, \quad B = \lim_{x \to 2} (x + 5) = 7. \]

(c) \[ \lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} = \lim_{x \to 5} \frac{(x - 5)(x + 1)}{x - 5} = \lim_{x \to 5} (x + 1) = 6. \]

(d) \[ \square \]

(e) \[ No \] because \[ \lim_{x \to 2} g(x) = 2 \cdot 2 - 1 = 7 \] and \[ \lim_{x \to 2} g(x) = 2 + 3 = 5 \]

\((or \quad \lim_{x \to 2} g(x) \text{ does not exist})\)

(f) \[ f' = -(5 - 2x)^2 (2 - 2) = \frac{2}{(5 - 2x)^3} \Rightarrow f''(2) = \frac{2}{(5 - 2 \cdot 2)^3} = 2. \] Hence, an equation of the tangent line is \[ y - 1 = 2(x - 2). \]

(g) \[ p'(t) = \frac{200(t + 2) - 200t}{(t + 2)^2} = \frac{400}{(t + 2)^2} \Rightarrow p'(5) = \frac{400}{7^2}. \]

(h) \[ \frac{dy}{dx} = 5e^{5x} \cos x - e^{5x} \sin x \]

(i) \[ \frac{dy}{dx} = \frac{1}{dx} (e^{6x} + e^{(x^6)}) = 6e^{6x} + e^{(x^6)} \cdot 6x^5. \]

(j) \[ \frac{d}{dx} (\tan(xy)) = \frac{d}{dx} (2x + y) \Rightarrow \sec^2(xy) \left( y + xy' \right) = 2 + y' \]
\[ \Rightarrow y' \sec^2(xy) + xy' \sec^2(y) = 2 + y' \Rightarrow (x \sec^2(xy) - 1)y' = 2 - y \sec^2(xy) \]
\[ \Rightarrow y' = \frac{2 - y \sec^2(xy)}{(x \sec^2(xy) - 1)} \]
(k) \[ F = 20000 \, e^{0.07t} \Rightarrow 8000 = 20000 \, e^{0.07t} \]
\[ \Rightarrow e^{0.07t} = 4 \Rightarrow 0.07t = \ln 4 \Rightarrow t = \frac{\ln 4}{0.07} \]

(l) Let \( x \) be the length of an edge at time \( t \).
Then the volume \( V = x^3 \).
\[ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} = 9x^2 \]
When \( x = 10 \), we get \[ \frac{dV}{dt} = 9(10^2) = 900 \, \text{cm}^3/\text{sec} \]

(m) \[ \frac{dy}{dx} = \frac{-1}{\tan^{-1} x} - \frac{1}{1 + x^2} \]

(n) \[ f' = 7x^6 e^{-x} + x^7 e^{-x}(-1) = x^6 e^{-x}(7 - x) \]
\[ f' = 0 \Rightarrow x = 0 \text{ is the unique critical point} \]
\[ f(-1) = (-1)^7 e^{-(-1)} = -e, \quad f(0) = 0, \quad f(5) = 5^7 e^{-5} \]
Hence, \[ A = 5^7 e^{-5}, \quad B = -e \]
2. \( f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{5-3x} - \frac{1}{5-3(2)}}{x - 2} \)

\[= \lim_{x \to 2} \frac{\frac{1}{5-3x} + \frac{1}{5-3(2)}}{x - 2} = \lim_{x \to 2} \frac{1 + 5-3x}{(x-2)(5-3x)} = \lim_{x \to 2} \frac{6 - 3x}{(x-2)(5-3x)} \]

\[= \lim_{x \to 2} \frac{-3(x-2)}{(x-2)(5-3x)} = \lim_{x \to 2} \frac{-3}{5-3x} = \frac{-3}{5-3(2)} = 3. \]

3. (i) (a) \( \text{No.} \)

(i) (b) \( \text{No.} \)

(iii) \( f' = (x+3)^1 + (x+3) \cdot 2(x+3) = (x+3)(3x+3) \)

\( f' = 0 \Rightarrow \text{the critical points are } -3 \text{ and } 1. \)

(ii) (d) \( f \text{ is increasing} \quad f' \begin{array}{c|c|c}
\theta & - & + \\
\rightarrow & -3 & 1 \end{array} \)

on \(( -\infty, -3 ) \) and \(( 1, \infty ) \) \( \text{or on } (-\infty, -3] \) and \( [1, \infty) \).

(iv) (e) \( f'' = 3x-3 + (x+3)^3 = 6x+6 \quad f'' = 0 \Rightarrow x = -1 \)

\( f \text{ is CU on } (-1, \infty) \) \( \text{or } [-1, \infty) \)

[4] (f) \begin{array}{c}
-3 & -1 & -1 \\
\downarrow & \nearrow & \searrow \\
\text{local max} & \text{occurs at } x = -3 \quad \text{IP} & \text{local min. occurs at } x = 1 \\
\end{array} \]
4. (b) \( \frac{d^2}{dp} = 10 e^{-1-p} (-1) - \frac{8}{(p+1)^2} \Rightarrow \\
3(p) = \frac{p}{8} \cdot \frac{d}{dp} = \frac{p}{10 e^{-1-p} + \frac{8}{(p+1)^2}} \\
\left( -10 e^{-1-p} - \frac{8}{(p+1)^2} \right) \\
\] 

[2] (b) \( E(1) = \frac{1}{10 + \frac{8}{2}} \left( -10 - \frac{8}{2^2} \right) = -\frac{12}{14} > -1 \\
\text{Hence, } p \uparrow \Rightarrow R \uparrow \\

[7] (c) \( R(p) = pp = 10p e^{-1-p} + \frac{8p^2}{p+1} \\
R'(p) = 10e^{-1-p} + 10pe^{-1-p} (-1) + 8 \cdot \frac{p+1 - p}{(p+1)^2} \\
= 10(1-p)e^{-1-p} + \frac{8}{(p+1)^2} \\
R''(p) = 10(-1)e^{-1-p} + 10(1-p)e^{-1-p} (-1) + \frac{8(-2)}{(p+1)^3} \\
\text{Hence, } R(1) = 14; R'(1) = 2; R''(1) = -12. \\
The second order Taylor ploy for the revenue function is \\
\[ R(1) + R'(1)(x-1) + \left( R''(1) \right) \left( x-1 \right)^2 = 14 + 2(x-1) - 6(x-1)^2. \]

[3] (d) \( \frac{d^g}{dt} = 10 e^{-1-p} \left( - \frac{dp}{dt} \right) + 8(-1) \frac{1}{(p+1)^2} \frac{dp}{dt} \\
\text{when } p = 1 \text{ and } \frac{d^g}{dt} = 2 \frac{dp}{dt} \text{ we get } \frac{dp}{dt} = -2 \text{ (dollars/week)} \)
The total cost of construction is given by

\[ 240 = 4(4xh) + 2x^2 + 3x^2 \]

or

\[ 240 = 16xh + 5x^2 \]

\[ \Rightarrow h = \frac{240 - 5x^2}{16x} \]

The volume \( V \) of the box is given by

\[ V = x^2h = x^2 \cdot \frac{240 - 5x^2}{16x} = \frac{240x - 5x^3}{16} \quad (x > 0) \]

\[ \frac{dV}{dx} = \frac{240 - 15x^2}{16} = 0 \Rightarrow x^2 = \frac{240}{15} = 16 \Rightarrow x = 4. \]

Since \( V \) is increasing on \((0, 4)\) and decreasing on \((4, \infty)\), \( V(4) \) is the maximal volume. When \( x = 4 \),

\[ h = \frac{240 - 5(4^2)}{16(4)} = \frac{5}{2} \]

Hence, the dimensions of the box of maximal volume are \( 4 \times 4 \times \frac{5}{2} \).
6. Since \( f(x) \) is cont., we have

\[
2018 = \lim_{x \to 2017} (f(x) + x) = \lim_{x \to 2017} f(x) + \lim_{x \to 2017} x = f(2017) + 2017
\]

\[
\Rightarrow \quad f(2017) = 1.
\]

Let \( g(x) = (2018x^2 - x^3) f(x) - 2017 \). We have

\[
g(0) = -2017 < 0
\]

\[
\]

\[
g(2018) = -2017 < 0
\]

By IVT, there exist \( a \) in \((0, 2017)\) and \( b \) in \((2017, 2018)\) such that

\[
g(a) = g(b) = 0,
\]

which implies that the line \( y = 2017 \) intersects the graph of \( y = (2018x^2 - x^3) f(x) \) at least twice.