The University of British Columbia
Final Examination - April 23, 2015
Mathematics 105
All Sections

Closed book examination

Last Name ____________ First _______ Signature ______________

Student Number ______ Section Number ______ Instructor ___________

Special Instructions:
No books, notes, or calculators are allowed. A formula sheet is included.

Senate Policy: Conduct during examinations
- Each examination candidate must be prepared to produce, upon the request
  of the invigilator or examiner, his or her UBC card for identification.
- Candidates are not permitted to ask questions of the examiners or
  invigilators, except in cases of supposed errors or ambiguities in examination
  questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the
  expiration of one-half hour from the scheduled starting time, or to leave during
  the first half hour of the examination. Should the examination run forty-five
  (45) minutes or less, no candidate shall be permitted to enter the examination
  room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with
  established rules for a given examination, which will be articulated by the
  examiner or invigilator prior to the examination commencing. Should dishonest
  behaviour be observed by the examiner(s) or invigilator(s), pleas of accident
  or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar
  practices, may be immediately dismissed from the examination by the
  examiner or invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise
      authorized;
  (b) purposely exposing written papers to the view of other candidates or
      imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other
      memory aid devices other than those authorized by the examiner(s) and;
  (e) using or operating electronic devices including but not limited to
      telephones, calculators, computers, or similar devices other than those authorized
      by the examiner(s); (electronic devices other than those authorized by the
      examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must
  hand in all examination papers, and must not take any examination material
  from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall
  into the traditional, paper-based method, examination candidates shall adhere
  to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions
  communicated by the examiner(s) or invigilator(s).

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1. **Short-Answer Questions**. Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find an equation of the plane parallel to the plane \( x - 3y + 2z = 1 \) passing through the point \((1, 0, -1)\).

Answer:

(b) Are the level curves of the paraboloid \( z = x^2 + y^2 \) lines, circles, parabolas, hyperbolas or ellipses?

Answer:

(c) Let \( f(x, y) = y^3 \cos(2x) \). Find \( \frac{\partial^2 f}{\partial x \partial y} \).

Answer:
(d) \( \sum_{k=1}^{4} f(1 + k) \cdot 1 \) is a left Riemann sum for a function \( f(x) \) on the interval \([a, b]\) with \( n \) sub intervals. Find the values of \( a \), \( b \) and \( n \).

Answer:
\[
a = \quad, \quad b = \quad, \quad n = \quad
\]

(e) Suppose \( \int_{2}^{3} f(x) \, dx = -1 \) and \( \int_{2}^{3} g(x) \, dx = 5 \). Evaluate \( \int_{2}^{3} (6f(x) - 3g(x)) \, dx \).

Answer:

(f) For the function \( f(x) = x^3 - \sin 2x \), find its antiderivative \( F(x) \) that satisfies \( F(0) = 1 \).

Answer:
(g) Evaluate \( \frac{d}{dx} \left( \int_0^{\sin x} (t^6 + 8) \, dt \right) \).

Answer:

(h) Evaluate \( \int \frac{dx}{\sqrt{x^2 + 25}} \).

Answer:

(i) Evaluate \( \int_0^{\frac{\pi}{2}} x \cos x \, dx \).

Answer:
(j) Evaluate $\int \cos^3 x \, dx$.

Answer:

(k) Evaluate the integral $\int_0^1 \frac{x^4}{x^5 - 1} \, dx$ or state that it diverges.

Answer:
(1) Evaluate \( \sum_{k=7}^{\infty} \frac{1}{3^k} \).

Answer:

(m) Solve the differential equation \( y'(t) = e^{\frac{t}{2}} \cos t \). You should express the solution \( y(t) \) in terms of \( t \) explicitly.

Answer:

(n) Find the limit of the sequence \( \left\{ \ln \left( \sin \frac{1}{n} \right) + \ln(2n) \right\} \).

Answer:
Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[12] 2. (a) Evaluate \( \int \frac{e^x}{(e^x + 1)(e^x - 3)} \, dx \).
2.(b) Evaluate \( \int_{2}^{4} \frac{x^2 - 4x + 4}{\sqrt{12 + 4x - x^2}} \, dx \).
3. Let $f(x, y) = (x - 1)^2 + (y + 1)^2$.

(a) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y)$ on the circle $x^2 + y^2 = 4$. A solution that does not use the method of Lagrange multipliers will receive no credit, even if the answer is correct.
3.(b) Find the maximum and minimum values of the function $f(x, y)$ over the region $R = \{(x, y) : x^2 + y^2 \leq 4\}$. 
4. A continuous random variable $X$ is given by the following probability density function

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}|x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the expected value $E(X)$ of the random variable $X$.

(b) Let $F(x)$ be the cumulative distribution function for the random variable $X$. Find $F(x)$ for $0 < x < 1$. 
[16] 5.

(a) Suppose that \( \frac{df}{dx} = \frac{x}{1 + 3x^3} \) and \( f(0) = 1 \). Find the Maclaurin series for \( f(x) \).

(b) Determine whether the series \( \sum_{n=2}^{\infty} \frac{n^2 + n + 1}{n^5 - n} \) converges or diverges.
5.(c) Determine whether the series \( \sum_{m=1}^{\infty} \frac{3m + \sin \sqrt{m}}{m^2} \) converges or diverges.

5.(d) Determine whether the series \( \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3} \) converges or diverges.
6. Suppose that the series \( \sum_{n=0}^{\infty} (1 - a_n) \) converges, where \( a_n > 0 \) for \( n = 0, 1, 2, 3, \ldots \).

(a) Determine whether the series \( \sum_{n=0}^{\infty} 2^n a_n \) converges or diverges.

(b) Find the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n x^n \).
Math 105 Final Exam (April 23, 2015)

1. (a) \((x-1) - 3(y-\theta) + 2(z+1) = 0\) or \(x - 3y + 2z = -1\).

(b) circles.

(c) \[
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( y^3 \cos 2x \right) \right) = \frac{\partial}{\partial y} \left( 3y^2 \cos 2x \right) = 6y^2 \sin 2x = -6y^2 \sin 2x.
\]

(d) \(a = 2, b = 6, n = 4\).

(e) 
\[
\int_2^6 (6-3g) \, dx = 6 \int_2^3 f \, dx - 3 \int_2^3 g \, dx
\]

\[
= 6 (1) - 3 \cdot 5 = -21
\]

(f) 
\[
F(x) = \int_1^x f(t) \, dt = \int_1^x (x^3 - \sin 2x) \, dx = \frac{x^4}{4} + \frac{1}{2} \cos 2x + C
\]

\[
f(2) = \frac{2^4}{4} + \frac{1}{2} \cos 2 \cdot 2 + C = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}.
\]

\[
f(x) = \frac{x^4}{4} + \frac{1}{2} \cos 2x + \frac{1}{2}
\]

(g) \((\sin x + 8) \cos x\).

(h) 
\[
\int \frac{dx}{\sqrt{x^2 + 25}} = \int \frac{5 \sec \theta \, d\theta}{5 \sec \theta + 25} = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln |\sqrt{\frac{x^2 + 25}{5}} + \frac{x}{5}| + C.
\]

(i) 
\[
\int_0^{\pi/2} \cos x \sin x \, dx = \left[ \frac{1}{2} \sin^2 x \right]_0^{\pi/2} = \frac{1}{2} \sin^2 \frac{\pi}{2} = \frac{1}{2}
\]

\[
\left[ \frac{1}{2} \sin x \right]_0^{\pi/2} = \frac{1}{2} \sin \frac{\pi}{2} = \frac{1}{2}
\]

\[
\left[ -\frac{1}{3} \sin^3 x \right]_0^{\pi/2} = -\frac{1}{3} \sin^3 \frac{\pi}{2} = -\frac{1}{3}
\]

\[
\int \cos^3 x \, dx = \int \cos x \cos^2 x \, dx = \int \cos x \left( 1 - \sin^2 x \right) \, dx = \int \cos x \, dx - \int \sin^2 x \cos x \, dx = \sin x - \frac{1}{3} \sin^3 x + C.
\]
1. (k) \[ \int_0^t \frac{x^4}{x^5-1} \, dx = \lim_{t \to 1^-} \int_0^t \frac{x^4}{x^5-1} \, dx \]

\[ = \lim_{t \to 1^-} \left( \frac{1}{5} \ln |x^5-1| \right) \bigg|_0^t = \lim_{t \to 1^-} \frac{1}{5} \ln |t^5-1| = -\infty, \]

so the integral is divergent.

(l) \( \frac{1}{8^2} \div (1 - \frac{1}{8}) \).

(m) \[ \frac{dy}{dt} = e^{-y/3} \cot \theta \implies \int e^{-y/3} \, dy = \cot \theta \, dt \]

\[ \int e^{-y/3} \, dy = \int \cot \theta \, dt = -3 e^{-y/3} = \sin \theta + C \]

\[ e^{-y/3} = \ln \left( -\frac{1}{2} \sin \theta - \frac{5}{2} \right) \implies y = -3 \ln \left( -\frac{1}{2} \sin \theta - \frac{5}{2} \right). \]

(n) \[ \lim \left( \ln \left( \sin \frac{1}{n} \right) + \ln 2n \right) = \lim \ln \left( 2n \sin \frac{1}{n} \right) \]

\[ = \lim \ln \left( 2 \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) = \ln 2. \]
2. (a) \[ \int \frac{e^x}{(e^{x+1})(e^x - 3)} \, dx = \frac{1}{u+1} \int du = e^x \, dx \]

\[ \frac{1}{u+1}(u-3) = \frac{A}{u+1} + \frac{B}{u-3} \Rightarrow 1 = A(u-3) + B(u+1) \]

\[ u= -1 \Rightarrow 1 = A(-4) \Rightarrow A = \frac{1}{4} \]

\[ u= 3 \Rightarrow 1 = B(4) \Rightarrow B = \frac{1}{4} \]

Hence,

\[ \int \frac{e^x}{(e^{x+1})(e^x - 3)} \, dx = \int \left( \frac{-1/4}{u+1} + \frac{1/4}{u-3} \right) \, du \]

\[ = -\frac{1}{4} \ln |u+1| + \frac{1}{4} \ln |u-3| + C = -\frac{1}{4} \ln |e^{x+1}| + \frac{1}{4} \ln |e^x - 3| + C \]

(b) \[ \int_2^4 \frac{\sqrt{x^2 - 4x + 4}}{x^2 + 4 - 4x} \, dx = \int_2^4 \frac{\sqrt{12 - (x^2 - 4x)}}{\sqrt{12 - (x^2 - 4x + 4)}} \, dx \]

\[ = \int_0^\pi \frac{(\sin \theta)^2}{\sqrt{16 - (\sin \theta)^2}} \, d\theta \]

\[ = 4 \int_0^\frac{\pi}{2} \sin \theta \, d\theta = 8 \left( \theta - \frac{1}{2} \cos \theta \right) \bigg|_0^{\frac{\pi}{2}} = 8 \left( \frac{\pi}{2} - \frac{1}{2} \cdot \frac{4}{2} \right) \]
3. (a) Let \( g(x, y) = x^2 + y^2 - 4 \). Then we need to solve the system:

\[
\begin{align*}
\frac{\partial f}{\partial x} = 2(3x) = 0 \\
\frac{\partial f}{\partial y} = 2(3y) = 0 \\
g(x, y) = x^2 + y^2 = 4
\end{align*}
\]

(1) \( y = \frac{x}{3y} \) \( \Rightarrow y = \frac{x}{3} \) (4)

(2) \( x = \frac{x}{2y} \) \( \Rightarrow x = \frac{x}{2} \) (5)

(4) and (5) \( \Rightarrow y = -x \) (6)

By (3) and (6), we get \( x^2 + (-x)^2 = 4 \Rightarrow x = \pm 2 \Rightarrow y = \mp 1 \).

\[f(2, -1) = (2^2 - 1)^2 + (-1 + 1)^2 = 2(3^2 - 1)^2 = 2(2 - 2\sqrt{2} + 1) = 6 - 4\sqrt{2} \]

\[f(-2, 1) = (-2^2 - 1)^2 + (2 + 1)^2 = 2(-3^2 + 1) = 2(2 + 2\sqrt{2} + 1) = 6 + 4\sqrt{2} \]

Hence, the maximum value of \( f(x, y) \) on the circle is \( 6 + 4\sqrt{2} \),

the minimum value of \( f(x, y) \) on the circle is \( 6 - 4\sqrt{2} \).

(b) \( \frac{\partial f}{\partial x} = 2(x-1) = 0 \)

\( \frac{\partial f}{\partial y} = 2(y+1) = 0 \) \( \Rightarrow (1, -1) \) is the only critical point.

\[f(1, -1) = (1-1)^2 + (-1+1)^2 = 0 \] (7)

By the conclusions in (a) and (7), we know that

the maximum value of \( f(x, y) \) on the region \( R \) is \( 6 + 4\sqrt{2} \),

the minimum value of \( f(x, y) \) on the region \( R \) is \( 0 \).
4. (a) \( F(x) = \int_{-\infty}^{\infty} x f(x) \, dx \)
\[ = \int_{-1}^{1} x \left( \frac{1}{4} - \frac{1}{2} x \right) \, dx + \int_{0}^{1} x \left( \frac{1}{4} + \frac{1}{2} x \right) \, dx \]
\[ = \int_{-1}^{0} \left( \frac{1}{4} x - \frac{1}{2} x^2 \right) \, dx + \int_{0}^{1} \left( \frac{1}{4} x + \frac{1}{2} x^2 \right) \, dx \]
\[ = \left[ \frac{1}{4} \frac{x^2}{2} - \frac{1}{2} \frac{x^3}{3} \right]_{-1}^{0} + \left[ \frac{1}{4} \frac{x^2}{2} + \frac{1}{2} \frac{x^3}{3} \right]_{0}^{1} \]
\[ = \left( \frac{1}{4} \frac{1}{2} - \frac{1}{2} \frac{1}{3} \right) + \left( \frac{1}{4} \frac{1}{2} + \frac{1}{2} \frac{1}{3} \right) \]
\[ = \left( \frac{1}{8} + \frac{1}{6} \right) + \left( \frac{1}{8} + \frac{1}{6} \right) = 0. \]

(b) \( F(x) = \int_{-\infty}^{\infty} f(t) \, dt = \int_{-\infty}^{-1} f(t) \, dt + \int_{-1}^{1} f(t) \, dt + \int_{1}^{\infty} f(t) \, dt \)
\[ = 0 + \int_{-1}^{0} \left( \frac{1}{4} - \frac{1}{2} t \right) \, dt + \int_{0}^{1} \left( \frac{1}{4} + \frac{1}{2} t \right) \, dt \]
\[ = \left( \frac{1}{4} t - \frac{1}{2} \frac{t^2}{2} \right)_{-1}^{0} + \left( \frac{1}{4} t + \frac{1}{2} \frac{t^2}{2} \right)_{0}^{1} \]
\[ = \left( \frac{1}{4} - \frac{1}{2} \frac{1}{2} \right) + \left( \frac{1}{4} + \frac{1}{2} \frac{1}{2} \right) \]
\[ = \left( -\frac{1}{4} - \frac{1}{4} \right) + \frac{1}{4} + \frac{1}{4} \frac{1}{2} \frac{1}{2} = \frac{1}{2} + \frac{1}{4} x + \frac{1}{4} x^2. \]
\[ f'(x) = x \cdot \frac{1}{1 - 3x^3} = x \sum_{n=0}^{\infty} (-1)^n 3^nx^{3n+1} \]
\[ f(x) = \left( \sum_{n=0}^{\infty} (-1)^n 3^nx^{3n+1} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^{3n+2}}{3n+2} \]

With \( f(1) = 1 \), we have \( C = 1 \) so
\[ f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+2} \]

(b) \[ \lim_{n \to \infty} \frac{(n^2 + n + 1) \sqrt{n^5 - n}}{n^5 + n^4 + n^3} = \lim_{n \to \infty} n^5 \frac{\sqrt{n^5 - n}}{n^5 - n} = 1 \]

Since \( \sum \frac{1}{n^3} \) converges,
\[ \sum_{n=0}^{\infty} \frac{n^2 + n + 1}{n^5 - n} \text{ converges.} \]

(c) Since \[ \frac{3m + \sin m}{m^2} > \frac{3m - 1}{m^2} \]
\[ \sum \frac{3m - 1}{m^2} \text{ diverges,} \quad \sum \frac{3m + \sin m}{m^2} \text{ diverges} \]

\[ \left( \text{or} \quad \frac{3m + \sin m}{m^2} > \frac{3m - m}{m^2} = \frac{2m}{m^2} = \frac{2}{m} \right) \]

(d) \[ f(x) = \frac{1}{x(\ln x)^3} \] is positive, decreasing and continuous for \( x \geq 2 \). Since
\[ \int_{2}^{\infty} f(x) dx = \int_{2}^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x(\ln x)^3} dx \]
\[ = \lim_{t \to \infty} \left[ \ln(\ln x)^{-3} \right]_{x=2}^{x=t} = \lim_{t \to \infty} \left( -\frac{1}{2u^2} \right) \bigg|_{u=\ln x}^{u=\ln t} = \frac{1}{2(u^2)} \]
\[ \text{du} = \frac{dx}{x} \text{ the series} \]
\[ \sum_{k=2}^{\infty} \frac{1}{k(k \ln k)^3} \text{ converges.} \]
b. (a) Since \( \sum_{n=0}^{\infty} (1-an) \) converges, \( \lim_{n \to \infty} (1-an) = 0 \) or \( \lim_{n \to \infty} an = 1 \). Hence, \( \lim_{n \to \infty} (2^n an) = \infty \), which implies that \( \sum 2^n an \) diverges.

(b) For \( |x| > 1 \), \( \lim_{n \to \infty} (\frac{\lim an \cdot x^n}{\lim an \cdot 1 \cdot x^n}) = \infty \). Hence, \( \lim an \cdot x^n \neq 0 \). This proves that \( \sum an \cdot x^n \) diverges for \( |x| > 1 \). (1)

Also, for \( |x| < 1 \), with \( x \neq 0 \), we have

\[
\lim \left| \frac{a_{n+1} \cdot x^{n+1}}{a_n \cdot x^n} \right| = \lim \left( \frac{a_{n+1}}{a_n} \cdot \frac{1 \cdot x}{1 \cdot x} \right) = \lim \frac{a_{n+1}}{a_n} \cdot |x| = |x| < 1
\]

\( \Rightarrow \sum an \cdot x^n \) converges for \( |x| < 1 \) (2)

By (1) and (2), the radius of convergence is 1.