Solutions to Math 184 Midterm 1 (Version 2)
(Oct. 17, 2019)

1. (a) \[ 5^0.4^{1/2} = 15^0, \quad 4^{1/2} = 3, \quad \frac{t}{12} \log_4 4 = \ln 3 \Rightarrow t = \frac{12 \ln 3}{\ln 4}. \]

Answer: \[ \frac{12 \ln 3}{\ln 4} \] or \[ \frac{12 \log_b 3}{\log_b 4} \] for any \[ b > 0 \text{ and } b \neq 1. \]

(b) \[ \lim_{x \to 4} \frac{x^2 - 16}{4 - x} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{-(x - 4)} = \lim_{x \to 4} (x + 4) = 8. \]

Answer: 8

(c) \[ \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (3x^2 - 2) = 3(-3)^2 - 2 = 25. \]

\[ \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (3x + 2b) = 3(-3) + 2b = -9 + 2b. \]

\[ \lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) \Rightarrow 25 = -9 + 2b \Rightarrow b = 17 \]

Answer: 17

(d) \[ f'(x) = \frac{1}{2} (4x + 1) \cdot 4 = 2 (4x + 1)^{-1/2} \]

\[ f'(2) = 2 (4 \cdot 2 + 1)^{-1/2} = 2 (9^{-1/2}) = \frac{2}{3}. \]

Answer: \[ 2 \left( 9^{-1/2} \right) \text{ or } \frac{2}{3} \]

(e) \[ f(x) = e^x + e^{x^2} \]

\[ f'(x) = e^x + 2xe^{x^2} \]

\[ f''(x) = 7e^x + 7xe^{x^2} \]

Answer: \[ 7e^x + 7xe^{x^2} \text{ or } \frac{2}{3} \]
\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{1}{4x - 1} - \frac{1}{4a - 1} \]

\[ = \lim_{x \to a} \frac{\left(\frac{1}{4x - 1} - \frac{1}{4a - 1}\right)(4x-1)(4a-1)}{(x-a)(4x-1)(4a-1)} \]

\[ = \lim_{x \to a} \frac{4a-1 - (4x-1)}{(x-a)(4x-1)(4a-1)} = \lim_{x \to a} \frac{4a-4x}{(x-a)(4x-1)(4a-1)} \]

\[ = \lim_{x \to a} \frac{-4(x-a)}{(x-a)(4x-1)(4a-1)} = \lim_{x \to a} \frac{-4}{(4x-1)(4a-1)} = \frac{-4}{(4a-1)^2}. \]
3. (a) Let \( p = f(q) = mq + b \). Since both \((1200, 5)\) and \((1140, 6)\) are on the demand curve, we get
\[
m = \frac{6 - 1}{1140 - 1200} = -\frac{1}{60}.
\]
Hence, the demand equation is given by
\[
p = -\frac{1}{60}q + 5.
\]
or
\[
p = -\frac{1}{60}q + 25.
\]

(b) The revenue function \( R(q) = pq = \left(-\frac{1}{60}q + 25\right)q = -\frac{1}{60}q^2 + 25q \), and the weekly cost function \( c(q) = 5q + 300\).

(c) The marginal weekly profit function is
\[
MP(q) = -\frac{1}{30}q + 20.
\]

(d) At a price of $4, the quantity sold per week is given by
\[
4 = -\frac{1}{60}q + 25,
\]
which implies that \( q = 1260 \). Since
\[
MP(1260) = -\frac{1}{30}(1260) + 20 < 0,
\]
the profit will increase if the price is increased (by a small amount).
4. (a) \[ h'(t) = 2 \cos(t^4) \cdot (-\sin(t^4)) \cdot 4t^3 \]
   \[ h'(0) = 0. \]
   Since \( h(0) = \cos^2 0 = 1 \), the equation for the tangent line at \( t = 0 \) is given by
   \[ y - 1 = h'(0) (t - 0) \quad \text{or} \quad y - 1 = 0. \]
   \[ \Rightarrow y = 1. \]

(b) Since the numerator is \( \ln(3x^2 - 2) - \ln 1 \),
   the limit is \( f'(1) \), where \( f(x) = \ln(3x^2 - 2) \).
   It follows that
   \[ \lim_{x \to 1} \frac{\ln(3x^2 - 2)}{x - 1} = f'(1) = \left( \frac{6x}{3x^2 - 2} \right) \bigg|_{x=1} = 6. \]
5. (b) \( y = f(x) = e^{2(x+1)} \Rightarrow x = e^{\frac{1}{2} \ln y - 1} = f^{-1}(y) \)

\[ \lim_{x \to 1} \frac{x^2 + 3 - 2}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x^2 + 3} - 2)(\sqrt{x^2 + 3} + 2)}{(x-1)(\sqrt{x^2 + 3} + 2)} \]

\[ = \lim_{x \to 1} \frac{x^2 + 3 - 2^2}{(x-1)(\sqrt{x^2 + 3} + 2)} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2 + 3} + 2)} \]

\[ = \lim_{x \to 1} \frac{x+1}{\sqrt{x^2 + 3} + 2} = \frac{1+1}{\sqrt{1^2 + 3} + 2} = \frac{2}{4} = \frac{1}{2} \]

(c) \( \) Let \( t = x^2 \). Then \( t \to 0^+ \) as \( x \to 0 \). Thus, we get

\[ A = \lim_{x \to 0} f(x^2) = \lim_{t \to 0^+} f(t) = 1 \]

Since \( \lim_{x \to 0^+} f(x^3) = \lim_{t \to 0^+} f(t) = 1 \), where \( t = x^3 \)

and \( \lim_{x \to 0^-} f(x^3) = \lim_{t \to 0^-} f(t) = -1 \). \( B = \lim_{x \to 0} f(x^3) \) does not exist.
6.15 The line $y = \frac{2019}{x} - 1$ intersects the graph of $y = f(x) = x^4 + 5x^3 + 2019x + 17$ if and only if the equation below has a solution:

$$x^4 + 5x^3 + 2019x + 17 = 2019x - 1$$

or

$$x^4 + 5x^3 + 18 = 0 \quad \cdots \quad (\star)$$

Let $g(x) = x^4 + 5x^3 + 18$. Since

$$g(-1) = (-1)^4 + 5(-1)^3 + 18 > 0,$$
$$g(-2) = (-2)^4 + 5(-2)^3 + 18 < 0$$

and $g(x)$ is continuous on $[-2, -1]$, the equation $(\star)$ has a solution in $(-2, -1)$.

Hence, the line $y = \frac{2019}{x} - 1$ intersects the graph of $y = f(x)$ at least once.