Solutions to Math 184 Midterm 1 (Version I)

(Oct. 17, 2019)

1. (a) [3] $100 \cdot 3^{t/15} = 200$, $3^{t/15} = 2$, $\frac{t}{15}\ln 3 = \ln 2 \Rightarrow t = \frac{15\ln 2}{\ln 3}$

$\boxed{\text{Answer: } \frac{15\ln 2}{\ln 3}}$ or $\boxed{\frac{15\log_b 2}{\log_b 3} \text{ for any } b>0 \text{ and } b\neq 1}$

(b) [3] $\lim_{x\to 5} \frac{x^2-25}{5-x} = \lim_{x\to 5} \frac{(x-5)(x+5)}{-(x-5)} = \lim_{x\to 5} [-(x+5)] = -10$

$\boxed{\text{Answer: } -10}$

(c) [3] $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (2x^2-3) = 2(-2)^2-3 = 5 \; (= f(-2))$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (2x+3b) = 2(-2)+3b = -4+3b$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) \Rightarrow 5 = -4+3b \Rightarrow b = 3$ $\boxed{\text{Answer: } 3}$

(d) [3] $f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = (2x+1)^{-1/2}$

$f'(4) = (2\cdot 4+1)^{-1/2} = 9^{-1/2} = \frac{1}{3}$

$\boxed{\text{Answer: } 9^{-1/2} \text{ or } \frac{1}{3}}$

(e) [3] $f(x) = e^{5x} + e^{(x^5)}$

$f'(x) = 5e^{5x} + 5x^4 e^{(x^5)}$

$\boxed{\text{Answer: } 5e^{5x} + 5x^4 e^{(x^5)}}$

2.(5) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{3x-1} - \frac{1}{3a-1}}{x - a}$

$$= \lim_{x \to a} \frac{\left(\frac{1}{3x-1} - \frac{1}{3a-1}\right)(3x-1)(3a-1)}{(x-a)(3x-1)(3a-1)}$$

$$= \lim_{x \to a} \frac{3a-1-(3x-1)}{(x-a)(3x-1)(3a-1)} = \lim_{x \to a} \frac{3a-3x}{(x-a)(3x-1)(3a-1)}$$

$$= \lim_{x \to a} \frac{-3(x-a)}{(x-a)(3x-1)(3a-1)} = \lim_{x \to a} \frac{-3}{(3x-1)(3a-1)} = \frac{-3}{(3a-1)^2}.$$

3. (a) [3] Let $p = f(q) = mq + b$. Since both $(1600, 200)$ and $(1520, 205)$ are on the demand curve, we get

$$m = \frac{205-200}{1520-1600} = \frac{5}{-80} = -\frac{1}{16}.$$

Hence, the demand equation is given by

$$p - 200 = -\frac{1}{16}(q - 1600)$$

or $p = -\frac{1}{16}q + 300$.

[2] (b) The revenue function $R(q) = pq = \left(-\frac{1}{16}q + 300\right)q = -\frac{1}{16}q^2 + 300q$, and the weekly cost function is $C(q) = 120q + 100000$.

[2] (c) The marginal weekly profit function is

$$MP(q) = \left(-\frac{1}{16}q^2 + 180q - 100000\right)' = -\frac{1}{8}q + 180.$$

[3] (d) At a price of \$250, the quantity sold per week is given by $250 = -\frac{1}{16}q + 300$, which implies that $q = 800$. Since

$$MP(800) = -\frac{1}{8}(800) + 180 > 0,$$

the profit will decrease if the price is increased (by a small amount).

4. (a) [3] $h'(t) = 2\cos(t^4) \cdot \left(-\sin(t^4)\right) \cdot 4t^3$

$h'(0) = 0.$

Since $h(0) = \cos^2 0 = 1$, the equation for the tangent line at $t=0$ is given by

$$y - 1 = h'(0)(t-0) \quad \text{or} \quad y - 1 = 0.$$

$$(\Rightarrow y = 1.)$$

(b) [3] Since the numerator is $\ln(3x^2-2) - \ln 1$, the limit is $f'(1)$, where $f(x) = \ln(3x^2-2)$. It follows that

$$\lim_{x\to 1} \frac{\ln(3x^2-2)}{x-1} = f'(1) = \left(\frac{6x}{3x^2-2}\right)\Bigg|_{x=1} = 6.$$

5. [3] (a) YES. $y = f(x) = e^{2(x+1)} \Rightarrow x = e^{2(y+1)}$

$\Rightarrow \ln x = \ln e^{2(y+1)} = 2y+2 \Rightarrow y = \frac{1}{2}\ln x - 1 = f^{-1}(x)$.

[3] (b) $$\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} = \lim_{x\to 1} \frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{(x-1)(\sqrt{x^2+3}+2)}$$

$$= \lim_{x\to 1} \frac{x^2+3-2^2}{(x-1)(\sqrt{x^2+3}+2)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+2)}$$

$$= \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+3}+2} = \frac{1+1}{\sqrt{1^2+3}+2} = \frac{2}{4} = \frac{1}{2}.$$

[3] (c) • Let $t = x^2$. Then $t \to 0^+$ as $x \to 0$. Thus, we get

$$A = \lim_{x\to 0} f(x^2) = \lim_{t\to 0^+} f(t) = 1.$$

• Since $\lim_{x\to 0^+} f(x^3) = \lim_{z\to 0^+} f(z) = 1$, where $z = x^3$

and $\lim_{x\to 0^-} f(x^3) = \lim_{z\to 0^-} f(z) = -1$. $B = \lim_{x\to 0} f(x^3)$

does not exist.

6. (5) The line $y = 2019x - 1$ intersects the graph of $y = f(x) = x^4 + 5x^3 + 2019x + 17$ if and only if the equation below has a solution:

$$x^4 + 5x^3 + 2019x + 17 = 2019x - 1$$

or $$x^4 + 5x^3 + 18 = 0 \quad \cdots\cdots \quad (*)$$

Let $g(x) = x^4 + 5x^3 + 18$. Since

$$g(-1) = (-1)^4 + 5(-1)^3 + 18 > 0,$$

$$g(-2) = (-2)^4 + 5(-2)^3 + 18 < 0$$

and $g(x)$ is continuous on $[-2, -1]$, the equation $(*)$ has a solution in $(-2, -1)$.

Hence, the line $y = 2019x - 1$ intersects the graph of $y = f(x)$ at least once.