Solutions to Math 184 Midterm 1 (Oct. 16, 2018)

(a) \[ f(x) = \frac{M}{2} \]

Answer: \( f(x) = \frac{M}{2} \)

(b) \[ \lim_{x \to a^+} f(x) = \frac{M}{2} \]

Answer: \( \frac{M}{2} \)

(c) \[ A = \lim_{x \to 3^-} f(x) = 3 \]
\[ B = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 5) = 3 + 5 = 8 \]
\[ C = \lim_{x \to 0} f(x) = \lim_{x \to 0} 3 = 3 \]

Answer: \( A = 3, B = 8, C = 3 \)

(d) \[ f''(x) = x^2 f'(x) + x f''(x) \]
\[ g'(x) = 2x f'(x) + 2 \cdot 2 f'(x) = 4(-1) + 4(3) = 8 \]

Answer: 8

(e) \[ \lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \to 25} \frac{(\sqrt{x} - 5)(\sqrt{x} + 5)}{(x - 25)(\sqrt{x} + 5)} = \lim_{x \to 25} \frac{x - 25}{(x - 25)(\sqrt{x} + 5)} = \lim_{x \to 25} \frac{1}{\sqrt{x} + 5} = \frac{1}{10} \]

Answer: \( \frac{1}{10} \)
2. \[ f''(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{(x - (-1))} = \lim_{x \to -1} \frac{\frac{3}{5x+2} - \frac{3}{5(-1)+2}}{x+1} \]

\[= \lim_{x \to -1} \frac{\frac{3}{5x+2} + 1}{x+1} = \lim_{x \to -1} \frac{\left(\frac{3}{5x+2} + 1\right)(5x+2)}{(x+1)(5x+2)} = \lim_{x \to -1} \frac{3 + (5x+2)}{(x+1)(5x+2)} \]

\[= \lim_{x \to -1} \frac{5x+5}{(x+1)(5x+2)} = \lim_{x \to -1} \frac{5(x+1)}{(x+1)(5x+2)} = \lim_{x \to -1} \frac{5}{5x+2} = \frac{5}{5(-1)+2} = \frac{5}{-3} = -\frac{5}{3}. \]

3. (a) \[ f(x) = \sin \left(\frac{\pi}{4} x\right) - x + \frac{1}{2}. \]

\[f(0) = \sin 0 - 0 - \frac{1}{2} > 0,\]

\[f(2) = \sin \left(\frac{\pi}{2}\right) - 2^2 + \frac{1}{2} = 1 - 4 + \frac{1}{2} < 0.\]

Since \( f(x) \) is continuous on \([0, 2]\), (By IVT), there is a solution to the equation \( \sin \left(\frac{\pi}{4} x\right) = x - \frac{1}{2} \) in the interval \((0, 2)\) (or \([0, 2]\)).

(b) \[ \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 + 3x + 2}{x+1} = \lim_{x \to -1} \frac{(x+1)(x+2)}{x+1} = \lim_{x \to -1} (x+2) = 1. \]

Hence, \( f(x) \) is continuous at \( x = -1 \) if and only if
\[\lim_{x \to -1} f(x) = f(-1) = 1. \]

Hence, we need to choose \( a \) such that \( a = 1 \).
4. \( f'(x) = \left( \frac{f(x)}{x-3} \right)' = \frac{f'(x)(x-3) - f(x)(x-3)'}{(x-3)^2} \)

\[ = \frac{f'(x)(x-3) - f(x)}{(x-3)^2} \]

\[ f'(2) = \frac{f'(2)(2-3) - f(2)}{(2-3)^2} = 3(-1) - 2 = -5 \]

(b) \( p(x) = (x-4)(x-c) \) for some constant \( c \)

\[ \text{(because } \lim_{x \to 4} p(x) = \lim_{x \to 4} \frac{p(x)}{x-4} (x-4) = \) \]

\[ = \lim_{x \to 4} p(x) - \lim_{x \to 4} (x-4) = 3 \cdot 0 = 0 \]

We get \( 3 = \lim_{x \to 4} \frac{p(x)}{x-4} = \lim_{x \to 4} \frac{(x-4)(x-c)}{x-4} \)

\[ = \lim_{x \to 4} (x-c) = 4-c \implies c = 1 \)

Hence, \( x^2 + ax + b = (x-4)(x-c) = (x-4)(x-1) = x^2 - 5x + 4 \)

\[ \implies a = -5 \text{ and } b = 4 \]
5. (a) Let \( P = f(q) = mq + b \) be the demand function, where \( m \) and \( b \) are constants. Since \((40, 14)\) and \((25, 17)\) are on the graph of \( P = f(q) \), we get

\[
m = \frac{17 - 14}{25 - 40} = \frac{3}{15} = \frac{1}{5}.
\]

Hence, we get

\[
P - 14 = -\frac{1}{5}(q - 40) \quad \text{or} \quad P = -\frac{1}{5}q + 22.
\]

(b) \[c(q) = 500 + 8q \]
\[R(q) = PQ = \left(-\frac{1}{5}q + 22\right)q = -\frac{1}{5}q^2 + 22q \]
\[P(q) = R(q) - c(q) = -\frac{1}{5}q^2 + 14q - 500.
\]

(c) The owner gets the maximal profit when

\[
q = -\frac{14}{2(-\frac{1}{5})} = 35.
\]

Hence, the price the owner should set is

\[
P(35) = -\frac{1}{5}(35) + 22 = 7 + 22 = 31.5.
\]

(d) If \( P = \$ \), we have \[10 = -\frac{1}{5}q + 22 \implies q = 60.
\]

Since \[
\frac{dR}{dq} = -\frac{2}{5}q + 22,
\]
we get \[
\frac{dR}{dq}\bigg|_{q = 60} = -\frac{2}{5}(60) + 22 < 0.
\]

Hence, we know that \( P \downarrow \implies R \downarrow \), i.e. if the price is decreased by a small amount, the weekly revenue decreases.
6. (a) \[ L = \lim_{h \to 0} \left( \frac{(4+h)^{2018} - 1}{h} + \frac{2(4+h)^{2017} - 2}{h} \right) \]

Here, we have two choices (at least):

Choice I: \[ f(x) = x^{2018}, \quad g(x) = 2x^{2017}, \quad c = 1 \]

Choice II: \[ f(x) = 2x^{2017}, \quad g(x) = x^{2018}, \quad c = 1 \]

(b) If you use the choice I, then

\[ f'(x) = 2018x^{2017}, \quad \frac{1}{2} g'(x) = \frac{1}{2} \cdot 2(2017)x^{2016} \]

\[ = 2018 (x)^{2017} + 2017(1)^{2016} = (1) \]

If you use the choice II, then

\[ f'(x) = 2(2017)x^{2016}, \quad g'(x) = 2018x^{2017} \]

\[ = 2(2017)(-1)^{2016} + \frac{1}{2} (2018)(1)^{2017} \]

\[ = 4034 + 3025 = 7059 \]