2-Categorial Derived Algebraic Geometry

Thesis Proposal

Robert Klinzmann

The proposal is organised as follows. In the first section we give an informal introduction into the ideas that lead to the development of derived algebraic geometry. The second section introduces the theory of derived schemes and stacks of Toën and Vezzosi. Kapranov's dg-schemes and a notion of Joyce's d-manifolds in an algebro-geometric context, called d-schemes, are described in section 4 and 5, respectively. They are examples of 2-categorial approximations of derived schemes. The last section is devoted to an outline of questions the author wants to answer in his thesis.

We always work over a field k of characteristic 0. Schemes and stacks are meant to be ordinary schemes and algebraic stacks over $\mathrm{Spec}(k)$. A differential graded k-algebra $A^\bullet = (A^\natural, d)$ is a graded k-algebra $A^\natural$ together with a morphism of graded k-algebras $d\colon A^\natural \to A^\natural$ of degree 1. Morphisms between differential graded k-algebras are morphisms of graded k-algebras that respect the differential. We will restrict to the category non-positively graded differential graded k-algebras, denoted by $\mathbf{cdga}_{\leqslant 0}$ and define $\mathbf{dgAff} := \mathbf{cdga}_{\leqslant 0}^{\mathrm{op}}$.

1 What is Derived Algebraic Geometry

We will give three examples in non-derived algebraic geometry that lead to derived algebraic geometry. The section is more informal and we do not claim that everything is well-defined and rigorous. Rather, it is a pool of ideas to motivate and study derived algebraic geometry.

1.1 Resolving Schemes

To understand quasicoherent sheaves on a scheme X nowadays one studies the derived category of quasicoherent sheaves, i. e., a sheaf is replaced by its injective resolution. Something similar can be done for the scheme itself. Let M be a smooth scheme, Y and Z two subschemes of M defined by ideal sheaves $\mathscr{I}$ and $\mathscr{J}$, respectively. Further, choose $x \in X := Y \cap Z := Y \times_M Z$. Then an intersection formula of Serre tells us that

$$m(Y,Z,M,x) = \sum_{i \geqslant 0} (-1)^i \cdot \mathrm{length}_{\mathscr{O}_{M,x}} \left(\mathrm{Tor}_i^{\mathscr{O}_{M,x}}(\mathscr{O}_{M,x}/\mathscr{I}_x, \mathscr{O}_{M,x}/\mathscr{J}_x) \right).$$

Here, we denote by $m = m(Y,Z,M,x)$ the intersection multiplicity of x in X inside M. Let us restrict to the case where $M = \mathrm{Spec}(C)$, $Y = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ are all local affine schemes. Then X is completely described by the local ring $R := A \otimes_C B$. From the formula above we see that m is not really an intrinsic part of R because we have to pick an injective resolution of R to compute the higher Tor's. To make m as intrinsic as possible, we really have to alter the tensor product by regarding A and B commutative differential graded C-algebras. We write $A^\bullet$ and $B^\bullet$, respectively. Since the tensor product of differential graded C-algebras is well-defined in the derived category, we get

$R^\bullet := A^\bullet \otimes^{\mathbf{L}}_C B^\bullet$ as the *true* tensor product of A and B. Since $\mathrm{H}^{-i}(R^\bullet) = \mathrm{Tor}_i^C(A, B)$ for all $i \geqslant 0$ and $\mathsf{length}(R^\bullet) = m$, this shows that $R^\bullet$ behaves more natural than R when it comes to questions in intersection theory.

Conclusion: Whatever derived algebraic geometry is, it is some kind of *algebraic geometry over commutative differential graded k-algebras.*

1.2 Hidden Smoothness Philosophy

The idea of *hidden smoothness* goes back to Kontsevich in his work [9, Sec. 1.4/1.5], where on the other hand he himself refers to a letter from Deligne to Esnault. It basically claims that singularities of moduli spaces arise due to truncations that we implicitly have performed during the construction of the parameter spaces. Let us illustrate this idea with the help of the following example.

Let S be a smooth projective complex algebraic surface. Then there exists an algebraic stack $\mathbf{Vect}_n(S)$ that classifies vector bundles of rank n on S. The fibre over a scheme X is simply given by

$$\mathbf{Vect}_n(S)(X) = \langle \text{vector bundles of rank } n \text{ on } X \times S \rangle$$

and morphisms are given by isomorphisms of vector bundles. Let us compute the fiber of the tangent stack of $\mathbf{Vect}_n(S)$ over a geometric point $E\colon \mathrm{Spec}(k) \to \mathbf{Vect}_n(S)$, which corresponds to a vector bundle of rank n over S, denoted by E as well. Let $D := \mathrm{Spec}(k[t]/(t)^2)$ be the scheme of dual numbers. By definition, $T_E\mathbf{Vect}_n(S)$ is the groupoid of morphisms $\widetilde{E}\colon D \to \mathbf{Vect}_n(S)$ that composed with the morphism $\mathrm{Spec}(k) \to D$, mapping t to 0, yields E. It is a basic fact in deformation theory, that morphisms $\widetilde{E}$ correspond to extensions of the vector bundle E over D and that these extensions can be classified by $\mathrm{Ext}^1_S(E, E) \simeq \mathrm{H}^1(S, \mathbf{End}(E))$. Automorphisms of an extension are described by $\mathrm{H}^0(S, \mathbf{End}(E))$. In addition to the groupoid structure, $T_E\mathbf{Vect}_n(S)$ is endowed with the structure of a Picard category, which roughly speaking means that $T_E\mathbf{Vect}_n(S)$ behaves like a vector space. Using a result of Deligne in [1, Expose XVIII], we can write $T_E\mathbf{Vect}_n(S)$ as

$$T_E\mathbf{Vect}_n(S) \simeq \tau^{\leqslant 0}(C^\bullet(S, \mathbf{End}(E))[1]),$$

where $C^\bullet(S, \mathbf{End}(E))$ is the trivial differential graded k-algebra with $C^i(S, \mathbf{End}(E)) = \mathrm{H}^i(S, \mathbf{End}(E))$. This suggests that the dimension if the tangent space is the truncated Euler characteristic:

$$\dim_{\mathbb{C}} T_E\mathbf{Vect}_n(S) = h^1(S, \mathbf{End}(E)) - h^0(S, \mathbf{End}(E)).$$

For a vector bundle E the Euler characteristic $\chi(C^\bullet(S, \mathbf{End}(E)))$ is locally constant. Since S is a surface, $\mathrm{H}^2(S, \mathbf{End}(E)) \neq 0$ in general. This implies that

$$\dim_{\mathbb{C}} T_E\mathbf{Vect}_n(S) = \chi(C^\bullet(S, \mathbf{End}(E))[1]) + h^2(S, \mathbf{End}(E))$$

cannot be locally constant. We can conclude that $\mathbf{Vect}_n(S)$ has singularities and as we have seen the reason for that lies in the fact that the tangent space is a truncation of a much longer complex.

The *invention* of new geometric objects that avoid the bad behaviour described above is one of the big leitmotifs of derived algebraic geometry. A plausible requirement for $\mathbf{RVect}_n(S)$ would be to require that the tangent space at a vector bundle E is not just the truncation but the whole complex $C^\bullet(S, \mathbf{End}(E))$:

$$T_E\mathbf{RVect}_n(S) \simeq C^\bullet(S, \mathbf{End}(E))[1].$$

But how does $\mathbf{RVect}_n(S)$ look like? We get a first idea by recalling that the total space of the tangent bundle of a smooth scheme is given by the spectrum of the symmetric algebra on the cotangent bundle. Putting this into the setting of our example above, we would like to have an equivalence

$$T\mathbf{RVect}_n(S) \simeq \operatorname{Spec} \operatorname{Sym}(T^\vee \mathbf{RVect}_n(S)).$$

The crucial point of our observation is, that $T^\vee\mathbf{RVect}_n(S)$, at least locally, is a complex of $\mathbb{C}$-modules living in degrees $(-\infty, 1]$. We will forget about the positive part in our observation because it comes from the stackiness of $\mathbf{Vect}_n(S)$. Thus, we are left with the insight that $\mathbf{RVect}_n(S)$ as a stack represents a moduli problem (functor) on the commutative objects in the category of non-positively graded complexes of $\mathbb{C}$-modules or equivalently non-positively graded differential graded $\mathbb{C}$-algebras.

Conclusion: If $\mathcal{M}$ is a good old 1-stack then the hidden smoothness idea predicts a *smooth derived stack* $\mathbf{R}\mathcal{M}$ defined on the category of commutative differential graded algebras together with a truncation $t_0(\mathbf{R}\mathcal{M})$ to a 1-stack that is equivalent to $\mathcal{M}$.

1.3 Simplicial structures

The previous two examples motivates us to study geometric objects over differential graded k-algebras. We now want to emphasise another important aspect of derived algebraic geometry. It is based on the observation that the category of commutative differential graded k-algebras is enriched over simplicial sets. This means that the set of morphisms $\operatorname{Mor}_{\mathbf{cdga}_{\leqslant 0}}(A^\bullet, B^\bullet)$ can be naturally extended to a simplicial set $\operatorname{Mor}^{\Delta}_{\mathbf{cdga}_{\leqslant 0}}(A^\bullet, B^\bullet)$.

Recall, that for an ordinary scheme X we have a natural bijection of sets

$$\operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec}(A), X) \cong \operatorname{Mor}_{\mathbf{CRng}}(\mathcal{O}_X(X), A)$$

for all commutative rings A. Thus, a scheme, when considered as its functor of points, is a presheaf of sets on commutative rings. It is reasonable to require an analogue relation in derived algebraic geometry using the fact that the set of morphisms can be extended to a simplicial set. For a derived scheme $\mathcal{X}$ this translates into a relation

$$\phi\colon \operatorname{Mor}(\operatorname{Spec}(A), \mathcal{X}) \longrightarrow \operatorname{Mor}^{\Delta}_{\mathbf{cdga}_{\leqslant 0}}(\mathcal{O}_{\mathcal{X}}(\mathcal{X}), A)$$

for all affine derived schemes $\mathrm{Spec}(A)$. We do not know yet what $\mathcal{X}$ or $\mathrm{Spec}(A)$ is, but the right hand side of the relation is a simplicial set and therefore encourages us to think about $\mathcal{X}$ as a presheaf on $\mathbf{cdga}_{\leqslant 0}^{\mathrm{op}}$ with values in simplicial sets. The relation ϕ is then required to be a weak equivalence of simplicial sets which is much more natural and useful than being an isomorphism.

Probably the more important reason to allow moduli functors with values in simplicial sets is that one wants to classify objects that admit non-trivial higher automorphisms. For instance, it is reasonable to define higher automorphism groups for objects in dg-categories or triangulated categories. To classify them, the notion of ordinary 1-stacks is insufficient since it was invented to keep track about 1-automorphisms only.

2 Derived Schemes and Stacks

We already used the notion *derived scheme* without knowing what it actually is. However, based on ideas and analogies to non-derived schemes, the functor of points of a derived scheme $\mathcal{X}$ should be a sheaf-like functor

$$\mathcal{X}\colon \mathbf{cdga}_{\leqslant 0}^{\mathrm{op}} \longrightarrow \mathbf{sSet}.$$

Here *sheaf-like* should indicate that we have a lot more strucutre to take into account than for sheaves of sets on a topological space.

The theory of derived schemes and stacks goes back to Toën and Vezzosi. They actually develop a very abstract but powerful theory of *algebraic geometry over model sites* in [16] and [17]. We will briefly describe basics of algebraic geometry over the model site $\mathbf{cdga}_{\leqslant 0}^{\mathrm{op}}$: derived schemes and stacks. Since almost everything is based on the idea of the functor of points, it seems to be more natural to define derived stacks first and then give conditions based on analogies in non-derived algebraic geometry for a derived stack to be a derived scheme.

Let $\mathrm{SPr}(\mathbf{dgAff})$ be the category of simplicial presheaves on $\mathbf{dgAff}$. In contrast to the functor of points of a scheme the domain and target of a simplicial presheaf $\mathcal{X} \in \mathrm{SPr}(\mathbf{dgAff})$ are endowed with non-trivial homotopic structure: $\mathbf{dgAff}$ and $\mathbf{sSet}$ are *model categories*.

A model category $\mathcal{M}$ is a category with three distinguished classes of morphisms – *weak equivalences*, *fibrations* and *cofibrations* – satisfying certain axioms. The main focus lies on the *homotopy category* $\mathrm{Ho}(\mathcal{M})$, which is formed by formally inverting weak equivalences. In general, it is almost impossible to study morphisms in the homotopy category. However, the extra data of fibrations and cofibrations resolve this problem and gives us a systematic tool to understand $\mathrm{Ho}(\mathcal{M})$. Since the homotopy category of $\mathcal{M}$ is equivalent to the homotopy category of its fibrant objects, the fibrant objects play an important role, very much like injectives or projectives in homological algebras. A last comment concerns $\mathcal{M}^{\mathrm{op}}$: it is a model category with a model structure induced by $\mathcal{M}$. The classes of cofibrations and fibrations are interchanged. For a detailed definition and treatment of model categories we refer to [7].

Let us briefly describe the model structures for **sSet** and $\mathbf{cdga}_{\leqslant 0}$ by defining weak equivalences and fibrations. We do not need to define cofibrations because they are determined by weak equivalences and fibrations by a certain lifting property.

- **sSet**: a morphism of simplicial sets $f\colon X \to Y$ is called a weak equivalence, if the geometric realisation of f is a weak homotopy equivalence of topological spaces. We call f a fibration if every diagram of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \overset{h}{\nearrow} & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

  has a lift h. Here, Λ^n_k denotes the simplicial set that arises by removing the interior and kth face of Δ^n.

- $\mathbf{cdga}_{\leqslant 0}$: a morphism $f\colon A \to B$ in $\mathbf{cdga}_{\leqslant 0}$ is called a weak equivalence if it is a quasiisomorphism, i. e., it induces an isomorphism $\mathrm{H}^i(f)\colon \mathrm{H}^i(A) \to \mathrm{H}^i(B)$ for all $i \leqslant 0$. We say f is a fibration if it is surjective in all degrees.

Let $f\colon A \to B$ be a morphism of commutative differential graded k-algebras. We say that f is *étale* if it induces an étale morphism of commutative k-algebras $\mathrm{H}^0(f)\colon \mathrm{H}^0(A) \to \mathrm{H}^0(B)$ and if the natural morphisms $\mathrm{H}^n(A) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^0(B) \to \mathrm{H}^n(B)$ are isomorphisms for all $n < 0$. A family of morphisms of differential graded k-algebras $\{A \to B_i\}_{i\in I}$ is called an *étale covering of* A if each $A \to B_i$ is étale and the induced family of affine schemes $\{\mathrm{Spec}(\mathrm{H}^0(B_i)) \to \mathrm{Spec}(\mathrm{H}^0(A))\}_{i\in I}$ is an étale covering of $\mathrm{Spec}(\mathrm{H}^0(A))$. One can show that this notion descends and defines a Grothendieck topology on $\mathrm{Ho}(\mathbf{cdga}_{\leqslant 0})$. A model category with a notion of covering that descents to a Grothendieck topology on its homotopy category is called a *model site*.

Before we go on, we very briefly explain *left Bousfield localisation*. Let $\mathcal{M}$ be a model category with weak equivalences W as well as S a class of morphisms in $\mathcal{M}$. We want to define a new model structure on $\mathcal{M}$ where elements in S become weak equivalences. In other words, we wish to formally invert morphisms in S in the homotopy category $\mathrm{Ho}(\mathcal{M})$. The technical gadget that does this for us is known as left Bousfield localisation. By some abstract nonsense, it will enlarge the class of weak equivalences such that morphisms in S belong to them, it downsizes fibrations and does not change cofibrations. We write $L_S\mathcal{M}$ for the new model structure.

Simplicial presheaves form a model category: weak equivalences and fibrations are defined objectwise. We need to refine this model structure for several reasons:

(1) It does not see the model structure on **dgAff**. In particular, a presheaf $F \in \mathrm{SPr}(\mathbf{dgAff})$ does not necessarily map a quasiisomorphism of commutative differential graded k-algebras to a weak equivalence of simplicial sets. But this

is certainly a condition we expect from a stack in this setting. To resolve this defect let $h_{(-)}\colon \mathbf{cdga}_{\leqslant 0} \to \mathrm{SPr}(\mathbf{dgAff})$ be the *constant Yoneda embedding*, i. e., for any commutative differential graded k-algebra A it is defined by

$$h_A := [B \longmapsto \mathrm{Mor}_{\mathbf{cdga}_{\leqslant 0}}(A,B)],$$

where the set of morphisms is considered as a constant simplicial set. Let W be the class of all morphisms $h_A \to h_B$ induced by weak equivalences $A \to B$. The category of *derived pre-stacks* is defined as

$$\mathbf{dgAff}^\wedge := L_W\,\mathrm{SPr}(\mathbf{dgAff}).$$

(2) Even the model structure $\mathbf{dgAff}^\wedge$ ignores the existence of the Grothendieck topology on $\mathrm{Ho}(\mathbf{cdga}_{\leqslant 0})$ or $\mathrm{Ho}(\mathbf{dgAff})$, respectively. This can be resolved by a further localisation. For a simplicial presheaf $F \in \mathbf{dgAff}^\wedge$ we can define on $\mathrm{Ho}(\mathbf{dgAff}^\wedge)$ *higher homotopy sheaves* $\pi_n(F,A,s)$ with $s \in F(A)_0$ on $\mathrm{Ho}(A/\mathbf{cdga}_{\leqslant 0})$. A morphism $f\colon F \to G$ of simplicial presheaves is a π_**-equivalence* or *étale local equivalence* if it induces isomorphisms on all higher homotopy sheaves. The model category of *derived stacks* is then defined to be the left Bousfield localisation

$$\mathbf{dgAff}^\sim := L_{\{\pi_*\text{-equivalences}\}}\mathbf{dgAff}^\wedge$$

One of the main results of Toën and Vezzosi in [16, Corollary 3.4.5] describes fibrant objects in $\mathbf{dgAff}^\sim$.

Theorem 2.1 *A simplicial presheaf $F \in \mathrm{SPr}(\mathbf{dgAff})$ is fibrant with respect to the model structure $\mathbf{dgAff}^\sim$ if and only if*

(1) *$F(A)$ is a fibrant simplicial set for any $A \in \mathbf{cdga}_{\leqslant 0}$.*

(2) *For any quasiisomorphism $A \to B$ in $\mathbf{cdga}_{\leqslant 0}$ the induced morphism of simplicial sets $F(A) \to F(B)$ is a weak equivalence.*

(3) *For any* étale hypercovering *$A \to B_*$ in $\mathbf{dgAff}$ the induced morphism*

$$F(A) \longrightarrow \mathrm{holim}_{[n]\in\Delta} F(B_n)$$

is a weak equivalence of simplicial sets.

The exact definition of hypercoverings is stated in [5] but for now it is enough to think about a hypercover as a co-nerve $A \to B_*$ induced by an étale cover $A \to B$, i. e., $B_n := B \otimes^{\mathbf{L}}_A \ldots \otimes^{\mathbf{L}}_A B$ is the n-fold tensor product of B over A. In this case condition (2) is more or less the homotopy analogue of the descent condition for ordinary stacks, the *homotopy sheaf condition.*

Definition 2.2 A product preserving simplicial presheaf $F \in \mathrm{SPr}(\mathbf{dgAff})$ satisfying condition (2) and (3) above is called a *derived stack*. A morphism of derived stacks is always considered as a morphism in $\mathrm{Ho}(\mathbf{dgAff}^\sim)$.

The category of derived stacks is denoted by $\mathbf{dSt} := \mathrm{Ho}(\mathbf{dgAff}^{\sim})$ and is a ∞-category because it is modelled on simplicial presheaves. By this we mean that the morphism set $\mathrm{Mor}(F,G)$ of two derived stacks can be naturally enriched to simplicial set $\mathrm{Map}(F,G)$ that behaves much like a topological space. For instance, $\pi_0 \, \mathrm{Map}(F,G) \cong \mathrm{Mor}(F,G)$.

A very important class of derived stacks are the *representable* ones. Every commutative differential graded k-algebra A defines a simplicial presheaf

$$\begin{aligned} \mathrm{Spec}(A)\colon \mathbf{dgAff} &\longrightarrow \mathbf{sSet} \\ B &\longmapsto \mathbf{cdga}_{\leqslant 0}(A, \Sigma_*(B)) \end{aligned}$$

in $\mathbf{dgAff}^{\sim}$. Here, $\Sigma_*(B)$ is a simplicial resolution of B. It is a non-trivial fact that $\mathrm{Spec}(A)$ is a derived stack, cf. [17, Lemma 1.2.12.3]. The above construction can be generalised to a functor $\mathbf{R}\mathrm{Spec}\colon \mathrm{Ho}(\mathbf{cdga}_{\leqslant 0}) \to \mathbf{dSt}$ on the homotopy categories. It turns out that this functor is fully faithful.

Definition 2.3 A derived stack $F \in \mathbf{dgAff}^{\sim}$ is *representable* if it is isomorphic in $\mathrm{Ho}(\mathbf{dgAff}^{\sim})$ to an object in the essential image of $\mathbf{R}\mathrm{Spec}$.

To draw an analogy to non-derived algebraic geometry, we use the name *affine derived scheme* instead of representable derived stack.

2.1 Geometry of Derived Stacks

The fact that the category of derived stacks is modelled by the model category $\mathbf{dgAff}^{\sim}$ implies the existence of *homotopy limits* and *homotopy colimits* in $\mathbf{dSt}$, cf. [6, Chapter 18, 19]. These are the homotopy analogues of limits and colimits in the framework of model categories. In particular, there exists homotopy fibred products, denoted by $\times^h$.

A morphism of derived stacks $f\colon F \to G$ is said to be *representable* if for any $\mathbf{R}\mathrm{Spec}(A) \to G$ the pullback $F \times^h_G \mathbf{R}\mathrm{Spec}(A)$ is isomorphic (in $\mathbf{dSt}$) to some affine derived scheme $\mathbf{R}\mathrm{Spec}(B)$. The notion of representability enables us to transfer a property $\mathbf{P}$ of a morphism of schemes to the derived world. We say that $f\colon F \to G$ has property $\mathbf{P}$ if f is representable and for any $\mathbf{R}\mathrm{Spec}(A) \to G$ the pullback morphism $\mathbf{R}\mathrm{Spec}(B) := F \times^h_G \mathbf{R}\mathrm{Spec}(A) \to \mathbf{R}\mathrm{Spec}(A)$ is induced by a morphism of commutative differential graded k-algebras $A \to B$ such that (1) $\mathrm{H}^n(A) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^0(B) \to \mathrm{H}^n(B)$ is an isomorphism for all $i < 0$ and (2) the morphism of affine schemes $\mathrm{Spec}(\mathrm{H}^0(B)) \to \mathrm{Spec}(\mathrm{H}^0(A))$ has property $\mathbf{P}$. In particular, we can speak about *flat, étale, smooth* and *Zariski open immersions.*

Definition 2.4 A *derived scheme* is a derived stack $\mathcal{X} \in \mathrm{Ho}(\mathbf{dgAff}^{\sim})$ such that

(1) X has a representable diagonal.

(2) There exists a family of Zariski open immersions $\{\mathbf{R}\mathrm{Spec}(A_i) \to X\}_{i\in I}$ such that the induced morphism
$$\coprod_i \pi_0(\mathbf{R}\mathrm{Spec}(A_i)) \longrightarrow \pi_0(X)$$
is an epimorphism of sheaves.

The ∞-category of derived schemes is denotes by **dSch**.

Recall, that a stack $\mathfrak{X}$ is algebraic if it admits a representable, smooth and surjective morphism $U \to \mathfrak{X}$ with U a scheme. Such a scheme U is called an atlas for $\mathfrak{X}$. We use this definition to define geometric derived stacks.

Definition 2.5 A derived stack $\mathcal{X} \in \mathbf{dSt}$ is *geometric* if the following conditions are satisfied:

(1) $\mathcal{X}$ has a representable diagonal.

(2) There exists a family of affine derived schemes $\mathbf{R}\mathrm{Spec}(A_i)$ together with smooth morphisms $\mathbf{R}\mathrm{Spec}(A_i) \to F$ such that
$$\coprod_i \pi_0(\mathbf{R}\mathrm{Spec}(A_i)) \longrightarrow \pi_0(F)$$
is an epimorphism of sheaves.

For a geometric derived stack F we can define a *cotangent complex* $\mathbf{L}_F$. For a point $x\colon \mathbf{R}\mathrm{Spec}(A) \to F$, the cotangent complex $\mathbf{L}_{F,x}$ is an element in the homotopy category of stable A-modules. It can be represented as a complex of k-modules concentrated in degree $(-\infty, 1]$. If $F = \mathbf{R}\mathrm{Spec}(B)$ is an affine derived scheme, then $\mathbf{L}_{F,x}$ is equivalent to the André-Quillen cotangent complex $L_{A/B}$. A derived stack F such that $\mathbf{L}_{F,x}$ has amplitude $[-1, \infty)$ for all possible points $x\colon \mathbf{R}\mathrm{Spec}(A) \to F$ is said to be *quasi-smooth.*

2.2 Derived Stack of Vector Bundles

To give at least one example of a derived stack that is not a classical stack we will describe briefly the construction of the derived stack of vector bundles of rank n on a smooth projective variety X. In the classical case for a given scheme T the T-points parametrise $\mathcal{O}_{X\times_k T}$-modules $\mathcal{E}$ flat over $\mathcal{O}_T$ such that for each closed point $t \in T$ the fibre $\mathcal{E}_t$ is a vector bundle of rank n on X.

To define a derived version we have to choose an affine derived stack $\mathbf{R}\mathrm{Spec}(A)$ instead of T. The analogue of a $\mathcal{O}_{X\times_k T}$-module is a dg-module $\mathcal{O}_X \otimes_k A$. Such a module $\mathcal{M}$ is said to be a vector bundle of rank n, if there exists an étale cover $\{U_i \to X\}_{i\in I}$ as well as an étale cover $\{A \to A_j\}_{j\in J}$ such that $\mathcal{M}(U_i) \otimes_A A_j$ is equivalent to $(\mathcal{O}_X|_{U_i} \otimes_k A_j)^n$. Here, a morphism of sheaves of dg-modules is defined to be an equivalence, if it induces weak equivalences on stalks. For any commutative differential graded k-algebra we denote by $w\mathbf{Vect}_n(X, A)$ the category of flat dg-modules $\mathcal{M}$ that are vector bundles of rank n together with weak equivalences as morphisms. Taking the nerve yields a simplicial set $\mathbf{RVect}_n(X)(A) := \mathcal{N}(w\mathbf{Vect}_n(X, A))$ and one can show that this defines a simplicial presheaf

$$\mathbf{RVect}_n(X)(-)\colon \mathbf{dgAff} \longrightarrow \mathbf{sSet}.$$

Theorem 2.6 [17, Cor. 1.3.7.12] *The simplicial presheaf $\mathbf{RVect}_n(X)$ is a geometric derived stack and its cotangent complex $\mathbf{L}_{\mathbf{RVect}_n(X),E}$ at a vector bundle E can be represented as the complex $C^\bullet(X, \mathbf{End}(E))[1]$.*

3 Kapranov's dg-Schemes

One of the very first approaches to derived algebraic geometry is the notion of dg-schemes by Kapranov introduced in [10] and [4]. In [9] Kontsevich called this kind of structure quasi-manifold.

Definition 3.1 A *dg-scheme* is a pair $X = (X^0, \mathcal{O}_X^\bullet)$, where X^0 is a k-scheme, $\mathcal{O}_X^\bullet$ is a sheave of commutative differential graded k-algebras such that $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$ and each $\mathcal{O}_X^i$ is a quasicoherent $\mathcal{O}_{X^0}$-module. A morphism $f\colon X \to Y$ of dg-schemes consists of a morphism $f_0\colon X^0 \to Y^0$ of schemes and a morphism $f^\bullet\colon f_0^{-1}\mathcal{O}_Y^\bullet \to \mathcal{O}_X^\bullet$ of sheaves of commutative differential graded k-algebras.

We denote the category of dg-schemes by **dgSch**. Every dg-scheme X has an underlying scheme $\pi_0(X) := \mathrm{Spec}(\mathrm{H}^0(\mathcal{O}_X^\bullet))$. Since the structure sheaf of a dg-scheme X is differentially graded, we can define its homotopy dg-scheme $X_h := (\pi_0(X), \mathrm{H}^\bullet(\mathcal{O}_X^\bullet))$, which is a dg-scheme with vanishing differentials. A morphism of dg-schemes $f\colon X \to Y$ is said to be a *weak equivalence* if f induces an isomorphism $f_h\colon X_h \to Y_h$.

A dg-scheme X is said to be *smooth* if X_0 is smooth and Zariski-local the structure sheaf is induced by a non-positively graded vector bundle, i. e., $\mathcal{O}_X^\bullet = \mathrm{Sym}_{\mathcal{O}_{X^0}}(E^\bullet)$ as sheaves of graded algebras.

A morphism $f\colon X \to Y$ of dg-schemes is defined to be *smooth* if $f_0\colon X^0 \to Y^0$ is a smooth morphism of schemes and Zariski-locally on X^0 we can write

$$\mathcal{O}_X^\bullet = \mathrm{Sym}_{\mathcal{O}_{X^0}}(E^\bullet) \otimes f_0^{-1}\mathcal{O}_Y^\bullet$$

as sheaves of graded algebras with $E^\bullet$ being a non-positively graded vector bundle on X.

dg-Schemes are very handy objects that in some cases are general but concrete enough to construct derived moduli spaces. We like to describe two examples in more detail.

3.1 The Derived Quot-Scheme

Let X be a projective scheme over a field k, $\mathcal{E}$ a coherent sheaf on X and Φ a polynomial with rational coefficients. Grothendieck constructed a scheme $\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$ that parametrises quotients of $\mathcal{E}$ with Hilbert polynomial Φ: a morphism of schemes $T \longrightarrow \mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$ corresponds to a pair $(\mathcal{F}, q)$ consisting of a coherent sheaf $\mathcal{F}$ on $X_T := X \times_k T$ flat over T together with a surjective $\mathcal{O}_{X_T}$-linear morphism $q\colon \mathcal{E}_T \to \mathcal{F}$ where $\mathcal{E}_T$ is the pull-back of $\mathcal{E}$ under the canonical projection $X_T \to T$. For a detailed construction we refer to [11].

In [4] Kapranov and Ciocan-Fontanine construct a dg-scheme $\mathbf{R}\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$ that satisfies the properties we expect from a derived version of $\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$. Namely, we recover the *hidden smoothness philosophy* described in Section (1.2):

(1) $\pi_0(\mathbf{R}\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}) \cong \mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$

(2) For every closed point $\mathcal{K} = [\mathcal{E}_k \to \mathcal{F}]$ the tangent space $T^{\bullet}_{\mathcal{K}}\mathbf{R}\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}$ is non-negatively graded and its cohomology given by

$$H^i(T^{\bullet}_{\mathcal{K}}\mathbf{R}\mathrm{Quot}^{\Phi}_{\mathcal{E}/X/\operatorname{Spec}(k)}) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}).$$

In general, the classical Quot-scheme is highly singular whereas the derived version $\mathbf{R}\mathrm{Quot}$ turns out to be a smooth dg-scheme as shown in [4, Theorem 4.2.1 and Definition 4.3.1].

3.2 The Derived Scheme of Actions

Let V be a finite dimensional k-vector space and A a unital associative k-algebra, not necessarily commutative. A morphism $\mu \in \mathrm{Hom}_k(A \otimes_k V, V)$ defines an A-action on V if and only if $\mu(a_1 \otimes \mu(a_2, b)) = \mu((a_1 a_2) \otimes v)$ for all $a_1, a_2 \in A$ and $v \in V$. The subspace of A-actions $W \subset \mathrm{Hom}_k(A \otimes_k V, V)$ induces a closed subscheme

$$\mathrm{Act}(A, V) := \operatorname{Spec}(\mathrm{Sym}_k(W^*)) \hookrightarrow \operatorname{Spec}(\mathrm{Sym}_k(\mathrm{Hom}_k(A \otimes V, V)^*))$$

which we call the A-action scheme of V.

The construction can be generalised to the case where A is a non-positively graded differential graded k-algebra. The vector space V is now considered to be graded and concentrated in degree 0. Then $A \otimes V$ and V are differential graded k-modules with $(A \otimes V)^n = A^n \otimes_k V$ and $d_{A \otimes V}(a \otimes v) = d_A(a) \otimes v$. Thus, $\mathrm{Hom}_{\mathrm{dg}}(A \otimes V, V)$ is a differential graded k-module. As in the non-graded case, the A-actions on V define a differential graded k-submodule $W \subset \mathrm{Hom}_{\mathrm{dg}}(A \otimes V, V)$ that induces a closed sub-dg-scheme

$$\mathrm{Act}_{\mathrm{dg}}(A, V) := \operatorname{Spec}(\mathrm{Sym}_k(W^*)) \hookrightarrow \operatorname{Spec}(\mathrm{Sym}_k(\mathrm{Hom}_{\mathrm{dg}}(A \otimes V, V)^*)).$$

The *derived scheme of A-actions* on V is defined to be

$$\mathbf{R}\mathrm{Act}(A, V) := \mathrm{Act}_{\mathrm{dg}}(Q(A), V),$$

where $Q(A) \to A$ is a *quasifree* resolution of A. In model category terms, $Q(A)$ should be a cofibrant replacement of A. In [4] Kapranov and Ciocan-Fontanine proved that $\mathbf{R}\mathrm{Act}(A, V)$ is smooth and that for a differential graded k-algebras A concentrated in degree 0

(1) $\pi_0(\mathbf{R}\mathrm{Act}(A, V)) = \mathrm{Act}(A, V)$ and

(2) for any $\mu \in \mathrm{Act}(A, V)$ the tangent space of $\mathbf{R}\mathrm{Act}(A, V)$ at μ is non-negatively graded and its cohomology is given by

$$\mathrm{H}^i(T^\bullet_\mu \mathbf{R}\mathrm{Act}(A, V)) = \begin{cases} T_\mu \,\mathrm{Act}(A, V) & i = 0 \\ \mathrm{Ext}^{i+1}_A(V, V) & i > 0 \end{cases}$$

On the level of closed points, the additional *derived information* in $\mathbf{R}\mathrm{Act}(A, V)$ parametrises A_∞-actions of A on V for A a non-positively graded differential graded k-algebra.

4 Joyce's d-Manifolds and d-Schemes

The algebraic model for schemes are commutative rings, i. e., the sheaf of regular functions is a sheaf of rings on X. The same is true for manifolds though the notion of ring is too coarse to distinguish manifolds by their sheaf of functions when considered as sheaves of rings. Instead, one is forced to use C^∞-rings. So in some sense a smooth manifold is a space M together with a sheaf of C^∞-rings. Based on this idea, Spivak defined in [13] the ∞-category of *derived manifolds*. Basically, a derived manifold is a topological space together with a sheaf of simplicial C^∞-rings that locally is the vanishing locus of a smooth function $\mathbf{R}^n \to \mathbf{R}^m$.

In the preprint [8] Joyce developed a theory of algebraic geometry over C^∞-rings. He uses this theory to define a 2-category of *d-manifolds* which in some sense can be seen as a 2-truncation of Spivak's derived manifolds. d-Manifolds are motivated by the observation that in enumerative geometry often the cotangent complex of moduli spaces is only concentrated in degrees $[-1, 0]$. This is certainly true for the very prominent moduli space of stable maps $\overline{\mathbf{M}}_g(S, \beta)$ for S a K3-surface.

We will briefly describe the notion of d-schemes, a modification of d-manifolds by replacing C^∞-schemes in Joyce's theory with ordinary schemes.

Definition 4.1 A *d-space* is a tuple $\mathfrak{X} = (X, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ consisting of

- schemes X and $X' := (X, \mathcal{O}'_X)$,
- a surjective morphism of sheaves $\imath_X \colon \mathcal{O}'_X \to \mathcal{O}_X$ such that $\mathcal{I}_X := \ker \imath_X$ is a quasicoherent $\mathcal{O}_X$-module with $\mathcal{I}_X^2 = 0$,
- a quasicoherent $\mathcal{O}_X$-module $\mathcal{E}_X$ and a surjective morphism $\jmath_X \colon \mathcal{E}_X \to \mathcal{I}_X$ of quasicoherent $\mathcal{O}_X$-modules.

From the definition we can deduce a notion of morphism between two d-spaces $\mathfrak{X} \to \mathfrak{Y}$. Such a morphism $\mathfrak{f}$ consists of a tuple (f, f', f'') where $(f, f^\natural)\colon X \to Y$ is a morphism of schemes, $f'\colon f^{-1}\mathcal{O}'_Y \to \mathcal{O}'_X$ a morphism of sheaves on X and $f''\colon f^*\mathcal{E}_Y \to \mathcal{E}_X$ a morphism of quasicoherent sheaves on X. These morphisms need to induce a commutative diagram

$$\begin{array}{ccccccc}
f^{-1}\mathcal{E}_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y = f^{-1}\mathcal{E}_Y & \xrightarrow{f^{-1}\jmath_Y} & f^{-1}\mathcal{O}'_Y & \xrightarrow{f^{-1}\imath_Y} & f^{-1}\mathcal{O}_Y & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{id}\otimes f^\natural} & & & & & & \\
f^{-1}\mathcal{E}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = f^*\mathcal{E}_Y & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f^\natural} & & \\
\downarrow{\scriptstyle f''} & & & & & & \\
\mathcal{E}_X & \xrightarrow[\jmath_X]{} & \mathcal{O}'_X & \xrightarrow[\imath_X]{} & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

We think about a d-space $\mathfrak{X}$ as a scheme X together with derived information given by $\mathcal{O}'_X$ and $\mathcal{E}_X$. The cotangent sheaf $\Omega_\mathfrak{X}$ should involve the derived information as well. It turns out that the 2-term complex $[\mathcal{E}_X \xrightarrow{\phi_X} \Omega_{X'}|_X]$ defined by the chain of morphism

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{I}_X \hookrightarrow \mathcal{O}'_X \xrightarrow{\delta} \Omega_{X'} = \Omega_{X'} \otimes_{\mathcal{O}'_X} \mathcal{O}_X \xrightarrow{\mathrm{id}\otimes\imath_X} \Omega_{X'}|_X$$

where δ denotes the universal derivation, gives the right definition for $\Omega_\mathfrak{X}$. We can think about the latter complex as an obstruction theory with obstruction sheaf $\mathcal{E}_X$ for the scheme X.

Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of d-spaces. Then f' induces a canonical morphism $f^\flat\colon f^*(\Omega_{Y'}|_Y) \to \Omega_{X'}|_X$ and f a canonical morphism $f^\sharp\colon f^*\Omega_Y \to \Omega_X$ making the diagram

$$\begin{array}{ccccccc}
f^*\mathcal{E}_Y & \xrightarrow{f^*\phi_X} & f^*(\Omega_{Y'}|_Y) & \xrightarrow{f^*\Omega_{\imath_Y}} & f^*\Omega_Y & \longrightarrow & 0 \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle f^\sharp} & & \\
\mathcal{E}_X & \xrightarrow[\phi_X]{} & \Omega_{X'}|_X & \xrightarrow[\Omega_{\imath_X}]{} & \Omega_X & \longrightarrow & 0
\end{array}$$

commutative. We say that $(f^\flat, f'')$ define a morphism $\Omega_\mathfrak{f}\colon f^*\Omega_\mathfrak{Y} \to \Omega_\mathfrak{X}$. If $\mathfrak{g}\colon \mathfrak{X} \to \mathfrak{Y}$ is another morphism and $g = f$ then a 2-morphism $M\colon f \Rightarrow g$ is a homotopy of the morphism of 2-term complexes $\Omega_\mathfrak{f} \sim \Omega_\mathfrak{g}\colon f^*\Omega_\mathfrak{Y} \to \Omega_\mathfrak{X}$.

Proposition 4.2 *d-spaces, morphisms of d-spaces and 2-morphisms of d-spaces define a strict 2-category* **dSp**.

One can show that in **dSp** there exists 2-fibred products. However, the construction is suprisingly technical and we decided to omit the construction. The underlying scheme of a fibred product of d-spaces is the fibred product of the underlying schemes of the d-spaces involved.

Every scheme X defines in a trivial way a d-space $F(X) = (X, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$. This construction is functorial and defines a functor $F\colon \mathbf{Sch} \to \mathbf{dSp}$. In general, it is not true that $F(X) \times_{F(Z)} F(Y) \cong F(X \times_Z Y)$. This is actually a good thing because if we think about $X \times_Z Y$ being an intersection then the whole point of the d-space $F(X) \times_{F(Z)} F(Y)$ is that it remembers roughly how the intersection happened. This is related to the motivation in Section (1.1).

Definition 4.3 A d-space $\mathfrak{X}$ is a *principle d-scheme* if it is equivalent in $\mathbf{dSp}$ to $F(X) \times_{F(Z)} F(Y)$ for schemes $X \to Z \leftarrow Y$. A d-space $\mathfrak{X}$ is called a *d-scheme* if locally it is equivalent to a principle d-scheme.

We denote by $\mathbf{d} - \mathbf{Sch}$ the 2-category of d-schemes.

5 Outline of Thesis

Derived schemes and stacks are by now the far most general objects to study derived algebraic geometry. They form an ∞-category. Recall that this means that the set of morphisms behaves very much like a homotopy type of a topological space. However, sometimes a derived moduli space induced by a classical moduli problem has very has simple homotopical structure, e. g., it is kind of n-truncated for n very small.

The first interesting case is $n = 2$, i. e., we are looking for derived algebro-geometric objects that form a 2-category. This is one of the main motivations of Joyce for the development of d-manifolds in the differential-geometric context and can be carried over to the slightly relaxed notion of d-schemes in the algebro-geometric context: d-schemes are certain 2-truncated derived schemes.

Kapranov's dg-schemes are also of 2-categorial nature because we are able to define a meaningful notion of homotopy between morphisms of dg-schemes $f, g\colon X \to Y$. However, the naïve idea that a homotopy of morphisms is a homotopy of morphisms of chain complexes is not appropriate in our setting because it is not preserved by non-additive functors. To define a correct notion we have to define polynomial differential forms on the n-simplex:

$$\Omega^\bullet(\Delta^n) = k[T_1, \dots, T_n, \mathrm{d}T_1, \dots, \mathrm{d}T_n] / \langle (\mathrm{d}T_i)^2 \rangle$$

where the differential maps T_i to $\mathrm{d}T_i$. A 1-homotopy $H\colon f \sim g$ between morphisms of affine dg-schemes $f, g\colon \operatorname{Spec}(B^\bullet) \to \operatorname{Spec}(A^\bullet)$ is then a morphism of differential graded k-algebras

$$H\colon A^\bullet \to B^\bullet \otimes_k \Omega^\bullet(\Delta^1) = B^\bullet[T] \oplus B^\bullet[T] \cdot \mathrm{d}T$$

such that evaluating at $T = 0$ and $T = 1$ yields f and g, respectively. This can be extended to sheaves. 1-Homotopies are also connected via 2-homotopies and so on.

Hence, homotopy classes of 1-homotopies are a natural candidate for 2-morphisms in the category of dg-schemes.

The main focus of my thesis will focus on the understanding of the relationship between 2-categorial and ∞-categorial approaches to derived algebraic geometry. The question I would like to examine in detail is

How much of derived algebraic geometry can be captured by 2-categorial approximations?

The most canonical choice of a 2-category of derived schemes from the point of view of Toën and Vezzosi is a 2-truncation of the ∞-category of derived schemes as described in Section (2). The objects of this category are derived schemes but instead of using the whole space of morphisms $\mathrm{Map}_{\mathbf{dSch}}(\mathcal{X}, \mathcal{Y})$ of derived schemes $\mathcal{X}$ and $\mathcal{Y}$ we only use the 1-truncated mapping spaces $\tau_{\leqslant 1} \mathrm{Map}_{\mathbf{dSch}}(\mathcal{X}, \mathcal{Y})$ as the category of morphisms. We denote this 2-category by **2dSch**.

Other examples of 2-categorial approaches are dg-schemes and d-schemes as presented in the previous section as well as Behrend's 2-category of differential graded schemes as described in [2] and [3]. Behrend noticed in [ibid] that for certain geometrical constructions a 2-categorial approach is sufficient. This is supported by the existence of the derived Quot- and Hilbert schemes of Kapranov and Ciocan-Fontanine in the setting of dg-schemes. However, it lacks the existence of an universal family or in other words, it is not a solution to a moduli problem in dg-schemes.

We want to compare 2-categorial approaches with the 2-category of derived schemes **2dSch**. In the case of d-schemes this is expressed by the following

Conjecture 5.1 *There exists a 2-subcategory $\mathfrak{S}$ in* **2dSch** *and a pair of adjoint functors*

$$\mathfrak{S} \underset{\iota}{\overset{\Psi}{\rightleftarrows}} \mathbf{d-Sch}$$

that describes d-schemes as certain special derived schemes in $\mathfrak{S}$.

In detail we would like to find satisfying answers to the following questions:

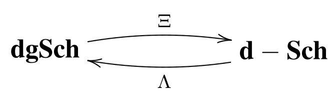
- What is the smallest or the most reasonable subcategory $\mathfrak{S}$ on that Ψ can be defined?
- How can we characterise objects in $\mathfrak{S}$?
- Assuming ι exists, is it full, faithful or at best even fully faithful?
- Can we describe the essential image of ι?
- How much information are we throwing away by passing from $\mathfrak{S}$ to $\mathbf{d-Sch}$ via Ψ?

- Is the information we forget relevant for moduli spaces in enumerative geometry, for instance $\overline{\mathbf{M}}_g(S,\beta)$?

For the construction of Ψ we believe that Pridham's characterisation of derived schemes as special simplicial affine dg-schemes (so called hypergroupoid in affine dg-schemes) can be very useful, cf. [12]. If $X_\bullet$ is such a simplicial affine dg-scheme, then the very rough idea is to define d-schemes $\mathfrak{X}_i$ for each X_i and to define glueing data for the $\mathfrak{X}_i$ that comes from the simplicial structure of $X_\bullet$. Since glueing is possible in $\mathbf{d-Sch}$ we end up with a d-scheme.

Yet another approach even for its own good is to understand the relationship between d-schemes and dg-schemes.

Conjecture 5.2 *There exists a pair of adjoint 2-functors*

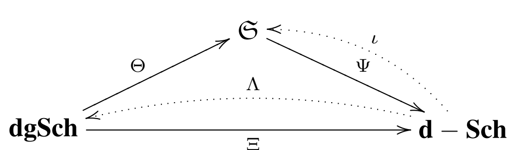


This conjecture also includes the claim that dg-schemes actually form a 2-category.

Let us explain why this result can be helpful. As described in [15, §3.3] we can define a 1-functor $\Theta\colon \mathrm{Ho}(\mathbf{dgSch}) \to \mathbf{dSch}$ that embeds the homotopy category of dg-schemes into the homotopy category of derived schemes. Actually, this functor is defined via a lift $\mathbf{dgSch} \to \mathbf{dSch}$ and it is very reasonable to expect that this defines a 2-functor $\mathbf{dgSch} \to \mathbf{2dSch}$. This functor is not well-studied but seems to be bad-behaved, e. g., there is some hard evidence that it is not full, cf. [14, §4.3 (2)]. The category of dg-schemes is in some sense too small. By this we mean that glueing dg-schemes with respect to weak equivalences is in general not possible inside $\mathbf{dgSch}$. Roughly speaking, to glue dg-schemes we have to keep track of higher homotopy information. If we use derived schemes instead of dg-schemes then gluing is possible. We believe that the category of derived schemes is somehow the smallest reasonable category where we can glue dg-schemes. If this is true, then we should be able to lift Ξ along Θ to a functor defined on some 2-subcategory inside $\mathbf{dSch}$. We hope that this category will turn out to be the conjectured 2-subcategory $\mathfrak{S}$ described above.

Finally, the proposal can be summarised by the conjectured existence of the following diagram of 2-functors.

Conjecture 5.3 *There exists a 2-subcategory $\mathfrak{S}$ in $\mathbf{2dSch}$ of certain derived schemes together with a 2-functor $\Psi\colon \mathfrak{S} \to \mathbf{d-Sch}$ making the diagram*



commutative. Furthermore, Ψ admits an adjoint $\iota\colon \mathbf{d-Sch} \to \mathfrak{S}$ that allows us to understand Joyce's d-schemes as derived schemes.

We have restricted ourselves to derived geometry with focus on schemes. However, dg-schemes and d-schemes can be generalised to dg-stacks and d-stacks, respectively. This should imply a similar picture and analogues questions as described above. This is important as lot of derived moduli spaces really have stacky structure and cannot be described just by derived schemes.

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University of British Columbia, 1984 Mathematics Road, Vancouver BC, V6T 1Z2, Canada

klinzmann@math.ubc.ca

http://www.math.ubc.ca/~klinzmann