4.2. Let $a, b \in \mathbb{Z}$, where $a \neq 0$ and $b \neq 0$. Prove that if $a|b$ and $b|a$ then $a = b$ or $a = -b$.

**Solution:** $a|b$ implies $b = ax$ for some $x \in \mathbb{Z}$ and $a \neq 0$. $b|a$ implies $a = by$ for some $y \in \mathbb{Z}$ and $b \neq 0$. So $a = by = axy$. Since $a \neq 0$, we have $1 = xy$. Then, since $x, y$ are integers, $x = y = 1$ or $x = y = -1$. Therefore, $a = b$ or $a = -b$.

4.10. Let $n \in \mathbb{Z}$. Prove that $2|(n^4 - 3)$ if and only if $4|(n^2 + 3)$.

**Solution:** Assume $2|(n^4 - 3)$. Then $n^4 - 3 = 2k$ for some integer $k$. Then $n^4 = 2k + 3$ is odd. By Theorem 3.12, then $n^2$ is odd, and then by Theorem 3.12 again $n$ is odd. So $n = 2a + 1$ for some integer $a$. Then $n^2 + 3 = 4k^2 + 4k + 1 + 3 = 4(k^2 + k + 1)$. This implies $4|(n^2 + 3)$ as $k^2 + k + 1$ is an integer.

Assume $2|(n^2 + 3)$. So $n^2 + 3 = 4b$ for some integer $b$. So $n^2 = 4b - 3$. Then $n^4 - 3 = (4b - 3)^2 - 3 = 16b^2 - 24b + 6 = 2(8b^2 - 12b + 3)$ is even as $8b^2 - 12b + 3$ is an integer. So $2|(n^4 - 3)$.

4.18. Let $m, n \in \mathbb{N}$ such that $m \geq 2$ and $m|n$. Prove that if $a$ and $b$ are integers such that $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.

**Solution:** $a \equiv b \pmod{n}$ implies $n|(a - b)$. So $a - b = nk$ for some integer $k$. Then $a = b + nk$. $m|n$ implies $n = mp$ for some integer $p$. Thus, $a = b + nk = b + mpk$. Then $a - b = m(pk)$. Then $m|(a - b)$. Therefore, $a \equiv b \pmod{m}$.

4.22. Let $n \in \mathbb{Z}$. Prove each of the statements (a) – (f).

(a) If $n \equiv 0 \pmod{7}$, then $n^2 \equiv 0 \pmod{7}$.
(b) If $n \equiv 1 \pmod{7}$, then $n^2 \equiv 1 \pmod{7}$.
(c) If $n \equiv 2 \pmod{7}$, then $n^2 \equiv 4 \pmod{7}$.
(d) If $n \equiv 3 \pmod{7}$, then $n^2 \equiv 2 \pmod{7}$.
(e) For each integer $n$, $n^2 \equiv (7 - n)^2 \pmod{7}$.
(f) For every integer $n$, $n^2$ is congruent to exactly one of 0, 1, 2 or 4 modulo 7.

**Solution:** (a) $n \equiv 0 \pmod{7}$ implies $n = 7k$ for some integer $k$, so $n^2 = 7(7k^2)$. Then $n^2 \equiv 0 \pmod{7}$ as $7k^2$ is an integer.

(b) $n \equiv 1 \pmod{7}$ implies $n = 1 + 7k$ for some integer $k$. Then $n^2 = 1 + 14k + 49k^2 = 1 + 7(2k + 7k^2)$. So $n^2 \equiv 1 \pmod{7}$.

(c) $n \equiv 2 \pmod{7}$ implies $n = 2 + 7k$ for some integer $k$. Then $n^2 = 4 + 28k + 49k^2 = 4 + 7(4k + 7k^2)$. So $n^2 \equiv 4 \pmod{7}$.
(d) \( n \equiv 3 \pmod{7} \) implies \( n = 3 + 7k \) for some integer \( k \). Then \( n^2 = 9 + 42k + 49k^2 = 2 + 7(1 + 6k + 7k^2) \). So \( n^2 \equiv 2 \pmod{7} \).

(e) \( n^2 - (7 - n)^2 = 7(2n - 7) \). So \( n^2 \equiv (7 - n)^2 \pmod{7} \).

(f) This is a result of the parts (a) to (e) using a proof with seven cases. If \( n \) is congruent to 0, 1, 2, 3 modulus 7 then the result is already proven in (a) to (d). If \( n \) is congruent to 4, 5 or 6 modulus 7, then \( 7 - n \) is congruent to 3, 2 or 1 modulus 7 and we use part (e) combined with the result from (d), (c) and (b).

4.28. Prove that if \( r \) is a real number such that \( 0 < r < 1 \), then \( \frac{1}{r(1-r)} \geq 4 \).

**Solution:** Note that because \( r(1-r) > 0 \), the inequality is equivalent to \( 4r(1-r) \leq 1 \) or \( 1 - 4r(1-r) \geq 0 \). But

\[
1 - 4r(1-r) = 1 - 4r + 4r^2 = (1 - 2r)^2 \geq 0
\]

for every \( r \).

4.32. (a) Recall that \( \sqrt{r} > 0 \) for every positive real number \( r \). Prove that if \( a \) and \( b \) are positive real numbers, then \( 0 < \sqrt{ab} \leq (a + b)/2 \). (The number \( \sqrt{ab} \) is called the geometric mean of \( a \) and \( b \), while \((a + b)/2\) is called the arithmetic mean or average of \( a \) and \( b \).)

(b) Under what conditions does \( \sqrt{ab} = (a + b)/2 \) for positive real numbers \( a \) and \( b \)? Justify your answer.

**Solution:** (a)

*Proof.* Let \( a, b \in \mathbb{R} \) so that \( a, b > 0 \). Then \((a - b)^2 \geq 0\), and so

\[
0 \leq a^2 + b^2 - 2ab \quad \text{add} \ 4ab \ \text{to both sides}
\]

\[
4ab \leq a^2 + b^2 + 2ab = (a + b)^2
\]

Since \( 4ab \) is positive, take the square-root of both sides to obtain (using the result in EQ1)

\[
0 < 2\sqrt{ab} \leq (a + b)
\]

Dividing by 2 gives the result.

(b) If \( \sqrt{ab} = (a + b)/2 \) then squaring both sides we get \( a^2 + b^2 + 2ab = 4ab \) or \( a^2 + b^2 - 2ab = 0 \) or \((a - b)^2 = 0 \) so \( a = b \).
4.34. Prove for every three real numbers $x, y$ and $z$ that $|x - z| \leq |x - y| + |y - z|$.

\[ \text{Solution:} \]

\textit{Proof.} Let $x, y, z \in \mathbb{R}$. Note that $x - z = (x - y) + (y - z)$ and so applying the triangle inequality gives

\[ |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| \quad \text{as required} \]

\[ \square \]

4.38. Let $a, b, x, y \in \mathbb{R}$ and $r \in \mathbb{R}^+$. Prove that if $|x - a| < r/2$ and $|y - b| < r/2$, then $|(x + y) - (a + b)| < r$.

\[ \text{Solution:} \text{ By theorem 4.17 of the book} \]

\[ |(x + y) - (a + b)| = |(x - a) + (y - b)| \leq |x - a| + |y - b| < \frac{r}{2} + \frac{r}{2} = r. \]

4.42. Let $A$ and $B$ be sets. Prove that $A \cap B = A$ if and only if $A \subseteq B$.

\[ \text{Solution:} \text{ First, assume } A \cap B = A. \forall x \in A, x \in A \cap B \text{ so } x \in B. \text{ Then } A \subseteq B. \]

Next, assume $A \subseteq B. \forall x \in A, x \in B$ because $A \subseteq B$. So $x \in A \cap B$. Therefore $A \subseteq A \cap B$. On the other hand, $\forall x \in A \cap B, x \in A$. So $A \cap B \subseteq A$. Thus $A \cap B = A$.

4.48. Let $A = \{n \in \mathbb{Z} : 2|n\}$ and $B = \{n \in \mathbb{Z} : 4|n\}$. Let $n \in \mathbb{Z}$. Prove that $n \in A - B$ if and only if $n = 2k$ for some odd integer $k$.

\[ \text{Solution:} \text{ First assume } n \in A \cap B. \text{ } n \in A \text{ implies } n = 2k \text{ for some integer } k. \]

$n \notin B$ implies $4 \nmid n$. Then $4 \nmid 2k$. This implies $k$ is odd (If $k$ were even, then $k = 2l$ for some integer $l$. Thus, $2k = 4l$ which is divisible by 4).

Next, assume $n = 2k$, $k$ is odd. So $2|n, n \in A$. $k$ is odd implies $k = 2l + 1$ for some integer $l$. Hence, $n = 4l + 2$ which is not divisible by 4. So $n \notin B$. So $n \in A - B$.

4.56. Let $A, B$ and $C$ be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.
Solution: \( \forall x \in (A - B) \cup (A - C), x \in A - B \) or \( x \in A - C \). If \( x \in A - B \), then \( x \in A \) and \( x \notin B \). Since \( B \cap C \subseteq B \), \( x \notin B \cap C \). Therefore, \( x \in A - (B \cap C) \).

If \( x \in A - C \), then \( x \in A \) and \( x \notin C \). Since \( B \cap C \subseteq C \), \( x \notin B \cap C \). Therefore, \( x \in A - (B \cap C) \). Thus, \( (A - B) \cup (A - C) \subseteq A - (B \cap C) \).

\( \forall x \in A - (B \cap C), x \in A, x \notin B \cap C \). If \( x \in B \) then \( x \notin C \). So \( x \in A - C \), hence \( x \in (A - B) \cup (A - C) \). If \( x \notin B \), then \( x \in A - B \). Therefore \( x \in (A - B) \cup (A - C) \). Thus, \( A - (B \cap C) \subseteq (A - B) \cup (A - C) \).

Question. Let \( A, B \) be two sets. Prove: If \( A \cap B = \emptyset \), then \( A = (A \cup B) - B \).

Solution: \( \forall x \in A, x \in A \cup B \). Since \( A \cap B = \emptyset \), \( x \notin B \). So \( x \in (A \cup B) - B \). We conclude: \( A \subseteq (A \cup B) - B \). On the other hand, \( \forall x \in (A \cup B) - B, x \in A \cup B \) and \( x \notin B \) implies \( x \in A \). Then \( (A \cup B) - B \subseteq A \). Thus, we have shown: \( A = (A \cup B) - B \).

4.58. Let \( A, B \) and \( C \) be sets. Prove that \( A \cap (B \cap C) = (A \cup B) \cap (A \cap C) \).

Solution:

Proof. Let \( A, B, C \) be sets. We use the distributive law and DeMorgan’s laws to establish the equivalence.

\[
A \cap (B \cap C) = A \cap \left[ \overline{B \cup C} \right] \\
= A \cap \left[ \overline{B} \cup \overline{C} \right] \\
= (A \cap \overline{B}) \cup (A \cap \overline{C}) \\
= (\overline{A \cup B}) \cup (\overline{A \cap C}) \\
= (\overline{A} \cup \overline{B}) \cap (\overline{A} \cap \overline{C}) \\
= (A \cup B) \cap (A \cap C)
\]

4.64. For sets \( A \) and \( B \), find a necessary and sufficient condition for \( (A \times B) \cap (B \times A) = \emptyset \). Verify that this condition is necessary and sufficient.

Solution: We will prove that \( (A \times B) \cap (B \times A) = \emptyset \) if and only if \( A \cap B = \emptyset \).

Proof. We prove the contrapositive of each implication.
Assume that $A \cap B \neq \emptyset$. Hence there exists $x$ so that $x \in A$ and $x \in B$. Hence $(x, x) \in A \times B$ and $(x, x) \in (B \times A)$. Hence $(x, x) \in (A \times B) \cap (B \times A)$ and so it is non-empty.

Now assume that $(A \times B) \cap (B \times A)$ is non-empty. Then there is some $(x, y)$ so that $(x, y) \in (A \times B)$ and $(x, y) \in (B \times A)$. This tells us that both $x \in A$ and $x \in B$. Hence $x \in A \cap B$ and so $A, B$ are not disjoint.

4.68. Let $A, B, C$ and $D$ be sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Solution:

Proof. Let $A, B, C, D$ be sets. We prove each inclusion in turn.

Let $(a, b) \in \text{LHS}$. Hence $(a, b) \in A \times B$ and $(a, b) \in C \times D$. Thus $a \in A$ and $a \in C$ and so $a \in (A \cap C)$. Similarly $b \in B$ and $b \in D$ so $b \in (B \cap D)$. Hence $(a, b) \in \text{RHS}$.

Now let $(a, b) \in \text{RHS}$. Hence $a \in A \cap C$ and $b \in B \cap D$. Hence $a \in A$ and $b \in B$ so $(a, b) \in A \times B$. Similarly, $a \in C$ and $b \in D$ so $(a, b) \in C \times D$. Thus $(a, b) \in \text{LHS}$.

So $\text{LHS} = \text{RHS}$ as required.