1. Let \( P(x, y) : x^2 + y^2 = 1 \) and \( Q(x, y) : x + y = 1 \) be open sentences on the domain \( \mathbb{Z}^2 \). Find (with explanation) the set of all \((x, y) \in \mathbb{Z}^2\) for which \( P(x, y) \iff Q(x, y) \) is true.

Solution: (Recall that \( P \iff Q \) is T if \( P, Q \) are both T or F.)

First of all, we note that \( P(x, y) \) is T for \((x, y) \in \{(0, \pm1), (\pm1, 0)\}\) and F for all other values of \((x, y) \in \mathbb{Z}^2\). On the other hand, \( Q(x, y) \) is T for all \((x, y)\) such that \( x = 1 - y \), and F for all other values of \((x, y) \in \mathbb{Z}^2\). Hence \( P(x, y) \iff Q(x, y) \) is T if either \((x, y) \in \{(1, 0), (0, 1)\}\) (as then \( P(x, y) \) and \( Q(x, y) \) are T), or if \((x, y) \notin \{(1 - y, y) : y \in \mathbb{Z}\} \cup \{(0, -1), (-1, 0)\}\) (as then \( P(x, y) \) and \( Q(x, y) \) are F).

In other words, \( P(x, y) \iff Q(x, y) \) is T for all \((x, y)\) in \( \{(1, 0), (0, 1)\} \cup (\mathbb{Z}^2 - (\{(1 - y, y) : y \in \mathbb{Z}\} \cup \{(0, -1), (-1, 0)\})) \).

(1 pt) for knowing when a biconditional is T, (1 pt) for when \( P(x, y) \) is T, (1 pt) for when \( Q(x, y) \) is T. (1 pt) for when \( P(x, y) \) and \( Q(x, y) \) are both T, (1 pt) for when \( P(x, y) \) and \( Q(x, y) \) are both F.

2. Let \( A \) be the set of circles in \( \mathbb{R}^2 \) centred at \((0, 0)\), and let \( B \) be the set of circles in \( \mathbb{R}^2 \) centred at \((1, 1)\). Let \( P(A, B) \) denote the open sentence \( A \) and \( B \) have exactly two points in common on the domain \( A \times B \). Consider the quantified statement

\[ S : \forall A \in A, \exists B \in B \text{ s.t. } P(A, B). \]

(a) State \( S \) in words.

(b) State \( \sim S \) symbolically (with all quantifiers on the left hand side; do not just simply write \( \sim S \)).

(c) State \( \sim S \) in words.

Solution:

(a) For any circle (in the plane) centred at \((0, 0)\), there is circle centred at \((1, 1)\) which intersects it at exactly two points.

(b) We have

\[ \sim S \equiv \exists A \in A \text{ s.t. } \sim (\exists B \in B \text{ s.t. } P(A, B)) \equiv \exists A \in A \text{ s.t. } \forall B \in B, \sim P(A, B) \]

(c) There is some circle centred at \((0, 0)\) which does not intersect any circle centred at \((1, 1)\) at exactly two points. Equivalently, there is some circle \( A \) centred at \((0, 0)\) so that for all circles \( B \) centred at \((1, 1)\), we have that \(|A \cap B| \neq 2\).

(1 pt) for (a), (2 pts) for (b), (1 pt) for (c).
Let $a \in \mathbb{R}$ and $f$ be a function. Consider the following sentence:

*For every positive number $\varepsilon$, there is a positive number $\delta$ such that for all numbers $x$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.*

Write the statement symbolically and then negate it (simplify until there are no $\sim$’s). Finally, state the negated statement in words.

**Solution:**

The statement is

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Negating the statement, we obtain

$$\exists \varepsilon > 0 \text{ s.t. } \sim (\exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

This is equivalent to

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \sim (\forall x \in \mathbb{R}, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$$

which in turn is equivalent to

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } \sim (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

Recall that, by Theorem 2.21(a), $\sim (P \Rightarrow Q) \equiv P \land (\sim Q)$. Hence the above statement is equivalent to

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } (|x - a| < \delta) \land (|f(x) - f(a)| \geq \varepsilon).$$

Thus the negation of the original statement can be stated as *There is some positive number $\varepsilon$ so that for any positive number $\delta$, there is a point $x$ so that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$. In other words, For some positive number $\varepsilon$, we have that $|f(x) - f(a)| \geq \varepsilon$ for points $x$ arbitrarily close to $a.*

(1 pt) for writing the statement symbolically, (4 pts) for “pushing through” the $\sim$, (1 pt) for stating the negation in words.

Let $a < b \in \mathbb{R}$. The Intermediate Value Theorem (IVT) states that

*For any function $f : [a, b] \to \mathbb{R}$, if $f$ is continuous then for any real number $y$ between $f(a)$ and $f(b)$, there is a real number $c \in [a, b]$ such that $f(c) = y$.*

Write the IVT symbolically and then negate it (simplify until there are no $\sim$’s). Finally, state the negated statement in words (which is, of course, a false statement).
For a function \( f \) on \([a, b]\), let \( I_f \) denote the set of \( y \) between \( f(a) \) and \( f(b) \). That is, put
\[
I_f = \{ y : (f(a) \leq y \leq f(b)) \lor (f(b) \leq y \leq f(a)) \}.
\]
Let \( C \) denote the set of continuous functions on \([a, b]\). (Recall that \( \forall x \in S, P(x) \equiv (x \in S) \Rightarrow P(x) \).) The IVT says that
\[
f \in C \Rightarrow (y \in I_f \Rightarrow \exists c \in [a, b], f(c) = y).
\]
(Recall that \( \sim (P \Rightarrow Q) \equiv P \land (\sim Q) \).) Negating this statement, we obtain
\[
(f \in C) \land \sim (y \in I_f \Rightarrow \exists c \in [a, b], f(c) = y).
\]
Equivalently,
\[
(f \in C \land y \in I_f) \land (\exists c \in [a, b], f(c) = y) \equiv (f \in C) \land (y \in I_f) \land (\forall c \in [a, b], f(c) \neq y).
\]
Hence the negation of the IVT states that \textit{There is a continuous function \( f \) on \([a, b]\) and some value \( y \) between \( f(a) \) and \( f(b) \) that \( f \) does not obtain on \([a, b]\).}

(1 pt) for writing the statement symbolically, (3 pts) for “pushing through” the \( \sim \),
(1 pt) for stating the negation in words.

5. Let \( n \in \mathbb{N} \). Prove that if \( |n - 1| + |n + 1| \leq 1 \), then \( |n^2 - 1| \leq 4 \).

\textbf{Solution:}

\textbf{Result.} Let \( n \in \mathbb{N} \). If \( |n - 1| + |n + 1| \leq 1 \), then \( |n^2 - 1| \leq 4 \).

\textbf{Proof.} Suppose that \( n \in \mathbb{N} \). Then \( n \geq 1 \), so that \( |n + 1| > 1 \). Since \( |n - 1| \) is always nonnegative for \( n \in \mathbb{N} \), it follows that \( |n - 1| + |n + 1| > 1 \) as well. The hypothesis of the result is false for all \( n \in \mathbb{N} \), and therefore the result is true.

\textit{This was an example of a vacuous proof.}

6. Prove that if \( x \) is an odd integer, then \( 9x + 5 \) is even.

\textbf{Solution:}

\textbf{Result.} If \( x \) is an odd integer, then \( 9x + 5 \) is even.

\textbf{Proof.} Suppose that \( x \in \mathbb{Z} \) is odd. Then there exists \( k \in \mathbb{Z} \) so that \( x = 2k + 1 \). Then \( 9(2k + 1) + 5 = 18k + 14 = 2(9k + 7) \). As \( 9k + 7 \) is an integer, we have shown that \( 9x + 5 \) is even.

7. Let \( S = \{1, 5, 9\} \). Prove that if \( n \in S \) and \( \frac{n^2 + n - 6}{2} \) is odd, then \( \frac{2n^3 + 3n^2 + n}{6} \) is even.
Solution:

Result. Let \( S = \{1, 5, 9\} \). If \( n \in S \) and \( \frac{n^2 + n - 6}{2} \) is odd, then \( \frac{2n^3 + 3n^2 + n}{6} \) is even.

Proof. For \( n = 1, 5, \) and \( 9, \) the quantity \( \frac{n^2 + n - 6}{2} \) evaluates to \(-2, 12, \) and \( 42, \) respectively, all of which are even. As the hypothesis is false for all \( n \in S, \) the result is true.

8. Let \( x \in \mathbb{Z}. \) Prove that \( 5x - 11 \) is even if and only if \( x \) is odd.

Solution:

Result. Let \( x \in \mathbb{Z}. \) Then \( 5x - 11 \) is even if and only if \( x \) is odd.

Proof. We prove the “if” direction first, using its contrapositive. If \( x \) is even, then there exists \( k \in \mathbb{Z} \) so that \( x = 2k. \) Then

\[
5x - 11 = 5(2k) - 11 = 10k - 12 + 1 = 2(5k - 6) + 1
\]

As \( 5k - 6 \) is an integer, \( 5x - 11 \) is odd.

Next we prove the “only if” direction, using a direct proof. Suppose that \( x \) is odd. Then there exists \( k \in \mathbb{Z} \) such that \( x = 2k + 1. \) Then

\[
5x - 11 = 5(2k + 1) - 11 = 10k - 6 = 2(5k - 3)
\]

As \( 5k - 3 \) is an integer, \( 5x - 11 \) is even.

9. Let \( S = \{2, 3, 4\} \) and let \( n \in S. \) Use a proof by contrapositive to prove that if \( \frac{n^2(n - 1)^2}{4} \) is even, then \( \frac{n^2(n + 1)^2}{4} \) is even.

Solution: Result. Let \( S = \{2, 3, 4\}, \) and let \( n \in S. \) If \( \frac{n^2(n - 1)^2}{4} \) is even, then \( \frac{n^2(n + 1)^2}{4} \) is even.

The contrapositive of the statement is: If \( \frac{n^2(n - 1)^2}{4} \) is odd, then \( \frac{n^2(n + 1)^2}{4} \) is odd.

Proof. We argue using the contrapositive. For \( n = 3 \) and \( 4, \) the quantity \( \frac{n^2(n + 1)^2}{4} \) evaluates to 36 and 100, respectively. These are both even numbers, and so the contrapositive holds vacuously.

For the remaining case, \( n = 2, \) the quantity \( \frac{n^2(n + 1)^2}{4} \) evaluates to 9, which is odd; and so, to verify the contrapositive, we must check the parity of \( \frac{n^2(n - 1)^2}{4} \)
for $n = 2$. The quantity $\frac{n^2(n-1)^2}{4}$ evaluates to 1 in this case, which is also odd. This verifies the result. □

10. Prove that if $n \in \mathbb{Z}$, then $n^2 - 3n + 9$ is odd.

**Solution:** Case 1: $n$ is odd. In this case, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$n^2 - 3n + 9 = (2k + 1)^2 - 3(2k + 1) + 9 = 4k^2 - 2k + 7 = 2(2k^2 - k + 3) + 1.$$  

Since $2k^2 - k + 3$ is an integer, $n^2 - 3n + 9$ is odd.

Case 2: $n$ is even. In this case $n = 2k$ for some $k \in \mathbb{Z}$. Then

$$n^2 - 3n + 9 = 4k^2 - 6k + 9 = 2(2k^2 - 3k + 4) + 1$$

is odd since $2k^2 - 3k + 4$ is an integer.

11. Prove the implication: if $m, n \in \mathbb{Z}$ are odd then $m^2 + 3n^2$ is even. What is the converse of the implication? Is the converse true or false? Justify your answer.

**Solution:** Since $m, n$ are odd, $m = 2a + 1, n = 2b + 1$ for some $a, b \in \mathbb{Z}$. Then

$$m^2 + 3n^2 = (4a^2 + 4a + 1) + (12b^2 + 12b + 3) = 2(2a^2 + 2a + 6b^2 + 6b + 2)$$

is even as $2a^2 + 2a + 6b^2 + 6b + 2$ is an integer.

Converse: If $m^2 + 3n^2$ is even then $m, n$ are odd. This is not true. Take $m = n = 0$. Then $m^2 + 3n^2 = 0$ is even, but $m, n$ are also even.