1.2 Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on $x$.

(a) $\{1, 2, 3\}$
(b) $\{0, 1, 2, 3\}$
(c) $\{-2, -1\}$
(d) $\{-2, 2, 3\}$

Solution:

(a) $\{x \in S \text{ s.t. } x \geq 1\}$
(b) $\{x \in S \text{ s.t. } x \geq 0\}$
(c) $\{x \in S \text{ s.t. } x < 0\}$
(d) $\{x \in S \text{ s.t. } |x| \geq 2\}$

1.4 Write each of the following sets by listing it elements within braces.

(a) $A = \{n \in \mathbb{Z} : -4 < n \leq 4\}$
(d) $D = \{x \in \mathbb{R} : x^2 - x = 0\}$
(e) $E = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

Solution:

(a) $A = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
(d) $D = \{0, 1\}$
(e) $E = \emptyset$

1.14 Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for

(a) $A = \{1, 2\}$
(b) $A = \{\emptyset, 1, \{a\}\}$
1.16 Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

**Solution:**

$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$. $\mathcal{P}(\mathcal{P}(\{1\})) = \mathcal{P}(\{\emptyset, \{1\}\}) = \{\emptyset, \emptyset, \{\emptyset\}, \{1\}, \emptyset, \{\{1\}\}, \{\emptyset,\{1\}\}\}$ — cardinality is 4.

1.22 Let $U = \{1, 3, \ldots, 15\}$ be the universal set, set $A = \{1, 5, 9, 13\}$ and $B = \{3, 9, 15\}$. Determine the following: (you can use a Venn Diagram to help you visualize the problem)

(a) $A \cup B$
(b) $A \cap B$
(c) $A \setminus B$
(d) $B \setminus A$
(e) $\bar{A}$
(f) $A \cap \bar{B}$

**Solution:** The sets are

$A \cup B = \{1, 3, 5, 9, 13, 15\}$
$A \cap B = \{9\}$
$A - B = \{1, 5, 13\}$
$B - A = \{3, 15\}$
$\bar{A} = \{3, 7, 11, 15\}$
$A \cap \bar{B} = A - B = \{1, 5, 13\}$

1.24 Give examples of three sets $A, B$ and $C$ such that $B \neq C$ but $B - A = C - A$.

**Solution:** Let $B = \{1, 2\}, C = \{1, 3\}, A = \{2, 3\}$. Then $B - A = \{1\} = C - A$.

1.38 For a real number $r$, define $A_r = \{r^2\}$, $B_r = [r - 1, r + 1]$ and $C_r = (r, \infty)$. For $S = \{1, 2, 4\}$, determine:
(a) $\bigcup_{\alpha \in S} A_{\alpha}$ and $\bigcap_{\alpha \in S} A_{\alpha}$
(b) $\bigcup_{\alpha \in S} B_{\alpha}$ and $\bigcap_{\alpha \in S} B_{\alpha}$
(c) $\bigcup_{\alpha \in S} C_{\alpha}$ and $\bigcap_{\alpha \in S} C_{\alpha}$

**Solution:**

(a) $\bigcup_{\alpha \in S} A_{\alpha} = \{1, 4, 16\}$, $\bigcap_{\alpha \in S} A_{\alpha} = \emptyset$
(b) $\bigcup_{\alpha \in S} B_{\alpha} = [0, 5]$, $\bigcap_{\alpha \in S} B_{\alpha} = \emptyset$
(c) $\bigcup_{\alpha \in S} C_{\alpha} = (1, \infty)$, $\bigcap_{\alpha \in S} C_{\alpha} = (4, \infty)$

1.42 For each of the following collections of sets, define a set $A_n$ for each $n \in \mathbb{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbb{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

(a) $\{[1, 2 + 1), [1, 2 + 1/2), [1, 2 + 1/3), \ldots\}$
(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$

**Solution:**

(a) Let $A_n = [1, 2 + 1/n)$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = [1, 3)$$
why is 3 excluded? — think

$$\bigcap_{n \in \mathbb{N}} A_n = [1, 2]$$

(b) Let $A_n = (1-2n/n, 2n)$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = (-2, \infty)$$

$$\bigcap_{n \in \mathbb{N}} A_n = (-1, 2)$$

1.54 Let $A = \{1, 2, \ldots, 12\}$. Give an example of a partition $S$ of $A$ satisfying the following requirements: (i) $|S| = 5$, (ii) there is a subset $T$ of $S$ such that $|T| = 4$ and $|\bigcup_{X \in T} X| = 10$ and (iii) there is no element $B \in S$ such that $|B| = 3$. 
**Solution:** Let $S = \{\{1\}, \{2\}, \{3,4,5,6\}, \{7,8,9,10\}, \{11,12\}\}$. Then $S$ is a partition of $A$ (it is a set of subsets of $A$) with cardinality 5. Also $S$ has no element of size 3. Partition $S$ would also satisfy condition (ii) as we can find a subset $T$ of $S$ with the union of its elements to have size 10, for example let: $T = \{\{1\}, \{2\}, \{3,4,5,6\}, \{7,8,9,10\}\}$, then $\bigcup_{X \in T} X = \{1,2,3,4,5,6,7,8,9,10\}$.

1.66 For $A = \{a \in \mathbb{R} : |a| \leq 1\}$ and $B = \{b \in \mathbb{R} : |b| = 1\}$, give a geometric description of the points in the $xy$-plane belonging to $(A \times B) \cup (B \times A)$.

**Solution:** Let $A = \{a \in \mathbb{R} : |a| \leq 1\} = \{a \in \mathbb{R} : -1 \leq a \leq 1\} = [-1,1]$, and $B = \{b \in \mathbb{R} : |b| = 1\} = \{-1,1\}$, then $A \times B$ is the set of all points $(x,y)$ on lines $y = 1$ or $y = -1$ with $x \in [-1,1]$, while $B \times A$ is the set of all points $(x,y)$ on lines $x = 1$ or $x = -1$ with $y \in [-1,1]$. Therefore, $(A \times B) \cup (B \times A)$ is the set of all points lying on (but not within) the square bounded by $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

1.74 For $A = \{1\}$ and $C = \{1,2\}$, give an example of a set $B$ such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.

**Solution:**

First work out the power sets

$$\mathcal{P}(A) = \{\emptyset, \{1\}\} \quad \mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

So let $B = \{\emptyset, \{1\}, \{2\}\}$. Then $A \subset B \subset C$ as required.

1.84 For $A = \{1,2,3\}$, let $B$ be the set of 2-element sets belonging to $\mathcal{P}(A)$ and let $C$ be the set consisting of the sets that are the intersections of two distinct elements of $B$. Determine $D = \mathcal{P}(C)$. 

**Solution:** Let's work out the sets

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\} \quad 2 \text{ element sets in } \mathcal{P}(A)$$

$$C = \{\{1\}, \{2\}, \{3\}\} \quad \text{intersections of elements of } B$$

So now we can finally work out $D$:

$$D = \mathcal{P}(C) = \{\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}\}$$