Chapter 2. Logic

**Defn** A logical statement (statement) is a sentence which can be assigned a truth value; it is either true or false (not both).

**Notation**
- Use $P, Q, R, \ldots$ for statements.
- Use indices if we have many statements.
- Use $T$ for value true and $F$ for value false.

**Ex** "3 is a prime integer" is a statement.

- $P$: "Square of an even integer, is an even integer" is a true statement.
- $Q$: "$\sqrt{2}$ is a rational number" is a false statement.
- $R$: $x^2 - 3x + 2 = 0$ is not a statement.
Ex "The 100th decimal digit of \( \pi \) is 7" is a statement.

Ex "The square of the length of the hypotenuse of a right-angle triangle is equal to the sum of the squares of the lengths of the other two sides."

This is a true statement.

Ex "Every even integer greater than 2 can be written as the sum of two primes."

"Goldbach's conjecture."

Logical Operators \((\neg, \lor, \land, \Rightarrow, \iff)\)

Negation Given statement \( P \) we can define a new statement "not \( P \)" called the negation of \( P \) denoted by \( \neg P \). \( \neg P \) is true if \( P \) is false. \( \neg P \) is false if \( P \) is true."
Disjunction. Given two statements \( P \) and \( Q \), we can define a new statement "\( P \ or \ Q \)" denoted by \( P \lor Q \) which is true if at least one of \( P \) and \( Q \) are true. \( P \lor Q \) is false if both \( P \) and \( Q \) are false.
Conjunction  Given two statements \( P \) and \( Q \) we can define a new statement "\( P \) and \( Q \)" denoted by \( P \land Q \) which is true if \( P \) and \( Q \) are both true, otherwise false.

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Ex  "\( 2 \) is an odd integer" and "\( \sqrt{2} \) is irrational" is a False statement.

Implication  Given two statements \( P \) and \( Q \) we can define a new statement "\( P \) implies \( Q \)" denoted by \( P \implies Q \) which is false if \( P \) is true and \( Q \) is false, otherwise it is true.
P is called the assumption of the conditional statement, Q is called the conclusion.

"P \Rightarrow Q"  P implies Q
if P then Q
Q if P
P is sufficient for Q
Q is necessary for P

Biconditional Given two statements P and Q we can define a new statement "P if and only if Q" or "P iff Q" or "P is a necessary and sufficient condition for Q" which is true if
P and Q have the same logical value otherwise it is false.

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**Compound Statement** is a statement with some logical connectors, for example $\neg P$, $P \land (\neg Q)$, $(Q \Rightarrow P) \lor R$

**Logical equivalence**

Two compound statements \( \forall \) are logically equivalent if they have the same logical values, for independent of logical \( \forall \)

\[ \begin{align*}
\text{Ex} & \quad \neg (\neg P) \equiv P \\
\text{Ex} & \quad (P \Rightarrow Q) \equiv (\neg P \lor Q)
\end{align*} \]

logically equivalent
Prove that \((P \Rightarrow Q) \equiv (\neg P \lor Q)\)

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Prove that \((P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)\)

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Defn Given a conditional statement \(P \Rightarrow Q\),
we define its converse as \(Q \Rightarrow P\)
and we define its contrapositive as \(\neg Q \Rightarrow \neg P\)
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Properties of disjunction, conjunction

**Commutative laws**

\[ P \lor Q \equiv Q \lor P \]
\[ P \land Q \equiv Q \land P \]

**Associative laws**

\[ (P \lor Q) \lor R \equiv P \lor (Q \lor R) \]
\[ (P \land Q) \land R \equiv P \land (Q \land R) \]

**Distributive laws**

\[ P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R) \]
\[ P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \]

**De Morgan's laws**

\[ \neg (P \lor Q) \equiv (\neg P) \land (\neg Q) \]
\[ \neg (P \land Q) \equiv (\neg P) \lor (\neg Q) \]
Quantified Statements

**Open sentence** is a sentence with one or more variables and its truth value depends on those variables.

\[
\text{Ex} \quad x^2 - 5x + 4 = 0 \\
\text{P}(x): \quad x^2 - 5x + 4 = 0
\]

\[
\text{Ex} \quad z - x^2 - y^2 + 3 = 0 \\
\text{Q}(x, y, z): \quad z - x^2 - y^2 + 3 = 0
\]

**Domain** The variables of an open sentence take values from some set \( D \), we call this the domain for the open sentence.

\[
\text{Ex} \quad \text{given open sentence} \quad \text{P}(x): \quad x^2 - 5x + 4 = 0
\]

over the domain \( S = \{0, 1, 2, 3, 4\} \)

decide for what values in the domain \( S \) is \( \text{P}(x) \) true, when is it false.

\[
\text{P}(0): \quad 0^2 - 5 \times 0 + 4 = 0 \quad \text{false statement}
\]

\[
\text{P}(1): \quad 1^2 - 5 \times 1 + 4 = 0 \quad \text{true statement}
\]

\[
\text{P}(2): \quad 2^2 - 5 \times 2 + 4 = 0 \quad \text{false}
\]

\[
\text{P}(3): \quad 3^2 - 5 \times 3 + 4 = 0 \quad \text{false}
\]

\[
\text{P}(4): \quad 4^2 - 5 \times 4 + 4 = 0 \quad \text{true statement}
\]
"For every \( x \in S \) we have \( P(x) \)."
this is a false statement, for example
\( 0 \in S \) and \( P(0) \) is false.

"There exists some \( x \in S \) such that \( P(x) \)
is true"
this is a true statement, for example
\( 1 \in S \) and \( P(1) \) is true.

Quantified Statements

Given open sentence \( P(x) \) and domain \( S \)
we can define two quantified statements:

"For all \( x \in S \) we have \( P(x) \)"
written as: \( \forall x \in S \; P(x) \)
this is a true statements if for every
\( x \in S \), \( P(x) \) is true statement
otherwise false.
we can define the statement:
"There exists some \( x \in S \) such that \( P(x) \)"
written as
\[
\exists x \in S \text{ s.t. } P(x)
\]
which is true if \( P(x) \) is true for at least one \( x \in S \)
otherwise false.

\[
\begin{align*}
\text{Ex} & \quad \mathcal{S} = \{0, 1, 2, 3, 4\} \\
\text{domain} & \quad P(x): x^2 - 5x + 4 = 0 \\
\text{open sentence} & \\
\end{align*}
\]

A: "\( \exists x \in S \text{ s.t. } P(x) \)" true statement
R: "\( \forall x \in S ; P(x) \)" false statement

\[
\begin{align*}
(\exists x \in S \text{ s.t. } P(x)) & \equiv P(0) \lor P(1) \lor P(2) \lor P(3) \lor P(4) \\
(\forall x \in S ; P(x)) & \equiv P(0) \land P(1) \land P(2) \land P(3) \land P(4)
\end{align*}
\]
Ex  Let \( P(n) \) be the open sentence "\( \frac{5n-6}{3} \) is an integer" over the domain \( \mathbb{Z} \). True or False? Why?

\( \forall n \in \mathbb{Z}, \ P(n) \)  
\( \exists n \in \mathbb{Z} \) s.t. \( P(n) \)

False for example \( P(1): \ \frac{5-6}{3} = \frac{-1}{3} \) is not an integer.

True for example \( P(0): \ \frac{5 \times 0 - 6}{3} = \frac{-6}{3} = -2 \) is an integer.

Non-existential

\[ \exists x \in S \text{ s.t. } P(x) \]
there does not exist any \( x \in S \) s.t. \( P(x) \)

\[ \exists! x \in S \text{ s.t. } P(x) \]
there exists exactly one \( x \in S \) s.t. \( P(x) \)

How to negate a quantified statement:

\( \neg (\forall x \in S, \ P(x)) \equiv (\exists x \in S \text{ s.t. } \neg P(x)) \)
\( \neg (\exists x \in S \text{ s.t. } P(x)) \equiv (\forall x \in S, \ \neg P(x)) \)
\[ \text{Ex} \quad \forall x \in \mathbb{R} \quad (x \in \mathbb{Q} \implies x^2 \in \mathbb{Q}) \quad \text{True statement} \]

\[ \neg \left( \forall x \in \mathbb{R} \quad (x \in \mathbb{Q} \implies x^2 \in \mathbb{Q}) \right) \]

\[ \equiv \exists x \in \mathbb{R} \quad \neg (x \in \mathbb{Q} \implies x^2 \in \mathbb{Q}) \equiv \exists x \in \mathbb{R} \quad x \in \mathbb{Q} \land x^2 \notin \mathbb{Q} \]

Remember:

\[ (p \implies q) \equiv (\neg p \lor q) \]

\[ \neg (p \implies q) \equiv p \land (\neg q) \]

\[ \text{Ex} \quad \exists y \in \mathbb{R} \quad \frac{1}{y} = y + 1 \quad \text{True statement} \]

\[ \frac{1}{y} = y + 1 \iff y^2 + y = 1 \iff y^2 + y - 1 = 0 \iff y = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \]

\[ \neg (\exists y \in \mathbb{R} \quad \frac{1}{y} = y + 1) \]

\[ \equiv \forall y \in \mathbb{R} \quad \frac{1}{y} \neq y + 1 \]
Ex True or false? Negate.

\[ \forall z \in \mathbb{Z} \exists w \in \mathbb{N} \text{ s.t. } z^2 < w \]

True, given any integer \( z \)
Let \( w = z^2 + 1 \) then \( w \in \mathbb{N} \) and \( w > z^2 \).

\[ \neg (\exists w \in \mathbb{N} \text{ s.t. } z^2 < w) \]
\[ \equiv \neg \exists z \in \mathbb{Z} \neg (\exists w \in \mathbb{N} \text{ s.t. } z^2 < w) \]
\[ \equiv \exists z \in \mathbb{Z} \text{ s.t. } \forall w \in \mathbb{N} \neg (z^2 < w) \]
\[ \equiv \exists z \in \mathbb{Z} \text{ s.t. } \forall z \in \mathbb{Z} \, , \, z^2 < w \]

False because given any natural \( w \)
we can find one integer \( z = w + 1 \)
then \( z^2 > w \).
Chapter 2. Logic

Defn: A sentence that can be assigned a truth value, is called a statement. Every logical statement is either true or false, but not both.

Use capital letters P, Q, R or \( P_1, P_2, P_3, \ldots \) for statements. Use T for true value and F for false.

Ex. P: "17 is a prime number" true statement

Q: "\( \sqrt{2} \) is a rational number" false statement

R: "The length of a side of a triangle is less than sum of the lengths of the other two sides" true statement

P: "The square of an even integer is an even integer." true statement
A sentence with one or more number of variable which its truth value depends on those variables is called an open sentence.

Ex: \( z = x^2 + y^2 - 1 \)

Ex: \( x^2 - 3x + 2 = 0 \)

Ex: "The 100th decimal of \( \pi \) is 7"

this is a statement.

Ex: "Every even integer greater than 2 can be written as sum of two primes."

Goldbach's conjecture.
Logical Operators \((\neg, \lor, \land, \Rightarrow, \leftrightarrow)\)

**Negation**
Given statement \(P\), we can define a new statement "not \(P\)" written as \(\neg P\) which is true if \(P\) is false and false if \(P\) is true.

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**Disjunction**
Given two statements \(P\) and \(Q\), we can define a new statement "\(P\) or \(Q\)" written as \((P \lor Q)\) which is true if at least one of the statements \(P\) or \(Q\) is true, is false if both \(P\) and \(Q\) are false.

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<th>(P)</th>
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<th>(P \lor Q)</th>
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Conjunction
Given statements $P$ and $Q$
we can define a new statement "$P$ and $Q$"
written as $P \land Q$ which is true if
both $P$ and $Q$ are true, otherwise false.

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<th>$P \land Q$</th>
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Implication
Given two statements $P$ and $Q$
we can define a new statement $P \Rightarrow Q$
read as "$P$ implies $Q$"
"if $P$ then $Q$"
"$Q$ if $P$"
"$P$ only if $Q$"
"$P$ is sufficient for $Q$"
"$Q$ is necessary for $P$"

$(P \Rightarrow Q)$ is false if $P$ is true and
Q is false, otherwise it is true.

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In conditional statement \((P \Rightarrow Q)\), \(P\) is called the assumption, \(Q\) is called the conclusion.

Biconditional \(\text{Given two statements } P \text{ and } Q\) we can define a new statement "\(P\) if and only if \(Q\)" denoted by "\(P \iff Q\)" read as "\(P\) iff \(Q\)" or "\(P\) is necessary and sufficient for \(Q\)"

"\(P \iff Q\)" is true if \(P\) and \(Q\) have the same logical values, otherwise it is false.

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A compound statement is a statement made of one or more statements with at least one logical operator.
Ex: \( \neg p, p \lor q, p \land (\neg q), (p \Rightarrow q) \lor r \)

Given two compound statements \( P \) and \( Q \) we say that they are logically equivalent or \( P \equiv Q \) if they have the same logical values independent of the logical values of their components.

Ex
Prove that \( (P \Rightarrow q) \equiv (\neg p \lor q) \)

\[
\begin{array}{c|c|c|c|c|c}
P & Q & P \Rightarrow q & \neg p & \neg p \lor q \\
T & T & T & F & T \\
T & F & F & F & F \\
F & T & T & T & T \\
F & F & T & T & T \\
\end{array}
\]
Ex Prove that \((P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)\)

\[
\begin{array}{c|c|c|c|c}
P & Q & P \Rightarrow Q & \neg Q & \neg P & \neg Q \Rightarrow \neg P \\
T & T & T & F & F & T \\
T & F & F & T & F & F \\
F & T & T & F & T & T \\
F & F & T & T & T & T \\
\end{array}
\]

Given a conditional statement \(P \Rightarrow Q\)
we define its converse as the new statement \(Q \Rightarrow P\)
and its contrapositive as the new statement \(\neg Q \Rightarrow \neg P\).

\[
\begin{array}{c|c|c|c}
P & Q & P \Rightarrow Q & Q \Rightarrow P \\
T & T & T & T \\
T & F & F & F \\
F & T & T & F \\
F & F & T & T \\
\end{array}
\]
Not related

Ex Prove that \((P \iff Q) \equiv (P \Rightarrow Q) \land (Q \Rightarrow P)\)

Ex \(\neg (\neg P) \equiv P\)

\[
\begin{array}{c|c|c}
P & \neg P & \neg (\neg P) \\
T & F & T \\
F & T & F \\
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\]
Proof of $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

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Proof of $\neg (P \lor Q) \equiv (\neg P) \land (\neg Q)$

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Ex. Prove that $\neg (P \Rightarrow Q) \equiv P \land (\neg Q)$

We know that $(P \Rightarrow Q) \equiv (\neg P \lor Q)$

So $\neg (P \Rightarrow Q) \equiv \neg (\neg P \lor Q)$

$\equiv \neg (\neg P) \land \neg (\neg Q)$

$\equiv P \land (\neg Q)$
**Ex**  Prove that

\[ \sim (p \iff q) \equiv (p \land \sim q) \lor (q \land \sim p) \]

**Proof**

\[ p \iff q \equiv (p \to q) \land (q \to p) \]

So

\[ \sim (p \iff q) \equiv \sim ((p \to q) \land (q \to p)) \]

\[ \equiv \sim (p \to q) \lor \sim (q \to p) \]

\[ \equiv (p \land \sim q) \lor (q \land \sim p) \]

**Ex**  Find the converse and the contrapositive of \((p \lor q) \to r\).

Converse: \(r \to (p \lor q)\)

Contrapositive: \((\sim r) \to (\sim (p \lor q))\)

\[ \equiv (\sim r) \to (\sim p \land \sim q) \]

\[ \equiv ((\sim r) \to (\sim p)) \land ((\sim r) \to (\sim q)) \]

**Ex**  Prove that \((p \to (q \land r)) \equiv (p \to q) \land (p \to r)\)

**Proof**

\[ p \to (q \land r) \equiv \sim p \lor (q \land r) \]

\[ \equiv \sim p \lor q \land \sim p \lor r \]

\[ \equiv (p \to q) \land (p \to r) \]
Properties of disjunction and conjunction

Given any two statements $P$ and $Q$:

\[ P \lor Q \equiv Q \lor P \]

commutative law

\[ P \land Q \equiv Q \land P \]

Associative law: Given statements $P$, $Q$, and $R$:

\[ (P \lor Q) \lor R \equiv P \lor (Q \lor R) \]
\[ (P \land Q) \land R \equiv P \land (Q \land R) \]

distributive law:

\[ P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R) \]
\[ P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \]

de Morgan's law:

\[ \neg (P \lor Q) \equiv (\neg P) \land (\neg Q) \]
\[ \neg (P \land Q) \equiv (\neg P) \lor (\neg Q) \]