Chapter 1. Sets

Defn A set is a collection of distinct objects, called the elements or members of the set.

Notation We use capital letters for sets $A, B, C, \ldots$ and small letters for the elements.

Remark There is no order or repetition in a set.

Examples

- $A = \{1, -\frac{\sqrt{2}}{3}, 3\}$ \hspace{1cm} |A| = 3
- $B = \{1, \{2, 3\}\}$ \hspace{1cm} |B| = 2
- $C = \{-2, -1, 0, 1, 2, \ldots, 10\}$ \hspace{1cm} |C| = 13

Notation Given set $A$ and object $a$, if $a$ is an element of $A$ we write $a \in A$, otherwise we write $a \notin A$. 
Definition: Empty set is the set with no elements, denoted by \( \emptyset \) or \( \{ \} \).

To describe a set, we can either list its elements or use the set-builder technique.

- \( A = \{ a \mid p(a) \} \) all \( a \) such that we have property \( p(a) \)
- \( A = \{ a \text{ s.t. } p(a) \} \)
- \( A = \{ a : p(a) \} \)

Example:

- \( C = \{ -2, -1, 0, 1, 2, \ldots , 10 \} \)
- \( C = \{ n \in \mathbb{Z} \mid -2 \leq n \leq 10 \} \)

Famous Sets in Math:

- \( \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \)
  set of natural numbers
- \( \mathbb{Z} = \{ \ldots , -2, -1, 0, 1, 2, \ldots \} = \{ 0, \pm 1, \pm 2, \ldots \} \)
  set of integers
Set of rationals
\[ \mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\} \]
\[ = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\} \]
\[ = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ and } \gcd(p, q) = 1 \right\} \]

Set of real numbers
\[ \mathbb{R} = \text{closure of set of rationals} \]
(real numbers are limits of sequences of rational numbers)

\[ \mathbb{I} = \left\{ r \in \mathbb{R} : r \notin \mathbb{Q} \right\} \]
set of irrational numbers.

\[ \mathbb{R}^2, \mathbb{R}^3, \ldots \] (we'll see later)

\[ \mathbb{P} = \left\{ 2, 3, 5, 7, 11, 13, \ldots \right\} \]
set of primes
(naturals with exactly two divisors)
**Defn** Given set $A$, by size of set $A$ denoted by $|A|$ or $\text{card}(A)$ we mean the number of distinct elements of $A$.

**Defn** Given two sets $A$ and $B$ we say that $A$ is a subset of $B$ and we write $A \subseteq B$ if every element of $A$ is an element of $B$.

**Defn** We say that set $A$ is a proper subset of set $B$ and we write $A \subset B$ if $A \subseteq B$ and $A \neq B$ meaning that every element of $A$ is an element of $B$ and there is at least one element in $B$ which is not in $A$.

**Ex**

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{N} \subseteq \mathbb{Z}$  $\mathbb{Z} \subseteq \mathbb{Q}$  $\mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{P} \subset \mathbb{N}$  $\mathbb{P} \subseteq \mathbb{N}$

$\emptyset \subseteq \mathbb{P}$  $\emptyset \subset \mathbb{P}$  $\mathbb{N} \subseteq \mathbb{N}$  $\mathbb{N} \not\subseteq \mathbb{N}$
Defn (Intervals of real numbers)

Let $a, b \in \mathbb{R}$ s.t. $a < b$; we define

$[a, b] = \{ r \in \mathbb{R} : a \leq r \leq b \} \subset \mathbb{R}$

$(a, b) = \{ r \in \mathbb{R} : a < r < b \} \subset \mathbb{R}$

$[a, b) = \{ r \in \mathbb{R} : a \leq r < b \}$

$(-\infty, b] = \{ r \in \mathbb{R} : r \leq b \}$

$(a, \infty) = \{ r \in \mathbb{R} : r > a \}$

$(-\infty, \infty) = \mathbb{R}$

$\pm \infty$ are not real numbers just a concept

Ex. Write down all subsets of set $S = \{1, 2\}$

$S_1 = \emptyset = \{\}$

$S_2 = \{1\}$

$S_3 = \{2\}$

$S_4 = \{1, 2\}$

$S_1 \subset S$ $S_1 \subset S$ $S_2 \subset S$ $S_3 \subset S$ $S_4 = S$ $S_4 \subset S$
**Defn (power set)** Given set $A$, the collection of all subsets of $A$ is called the power set of $A$ denoted by $\mathcal{P}(A)$ or $2^A$.

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

**Ex**

$S = \{1, 2\}$

$$\mathcal{P}(S) = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \}$$

$|S| = 2 \implies |\mathcal{P}(S)| = 2^2 = 4$

**Remark**

$|\mathcal{P}(S)| = 2^{|S|}$ (we'll prove this later)

**Ex**

$\mathcal{P}(\emptyset) = \{ \emptyset \}$

$|\emptyset| = 0 \implies |\mathcal{P}(\emptyset)| = 2^0 = 1$

**Ex**

$B = \{ \sqrt{2} \}$

$$\mathcal{P}(B) = \{ B_1, B_2 \} = \{ \emptyset, \{ \sqrt{2} \} \}$$

$|B| = 1 \implies |\mathcal{P}(B)| = 2^1 = 2$
Example

\[ C = \{ \emptyset, 1, \{0\} \} \]

Write all subsets of \( C \):

\[ C_1 = \{ \} = \emptyset \]
\[ C_2 = \{ \emptyset \} \]
\[ C_3 = \{ 1 \} \]
\[ C_4 = \{ \{0\} \} \]
\[ C_5 = \{ \emptyset, 1 \} \]
\[ C_6 = \{ \emptyset, \{0\} \} \]
\[ C_7 = \{ 1, \{0\} \} \]
\[ C_8 = C = \{ \emptyset, 1, \{0\} \} \]

\( \mathcal{P}(C) = \{ C_1, C_2, \ldots, C_8 \} \)

\( = \{ \emptyset, \{ \emptyset \}, \{1\}, \{\{0\}\}, \{\emptyset, 1\}, \{\emptyset, \{0\}\}, \{1, \{0\}\}, \{\emptyset, 1, \{0\}\} \} \)

\( |C| = 3 \quad |\mathcal{P}(C)| = 2^3 = 8 \)
Chapter 1: Sets

Defn: A set is a collection of distinct objects called the elements or members of the set.

There is no repetition or order in a set.

Notation: We use capital letters for sets and small letters for their elements.

Given set $A$ and element $a$, we say that $a \in A$ if $a$ is an element of $A$; otherwise we write $a \not\in A$.

Examples:

$A = \{ 1, 2, 3, 4 \}$  \hspace{1cm} |$A$| = 4

$B = \{ -\sqrt{2}, \frac{2}{3}, \{1\} \}$  \hspace{1cm} |$B$| = 3

$\sqrt{2} \not\in B$  \hspace{1cm} $-\sqrt{2} \in B$  \hspace{1cm} $1 \not\in B$

$\{1\} \in B$
**Defn** Size or cardinality of a set $A$ is the number of distinct elements in that set denoted by $|A|$ or $\text{card}(A)$.

**Ex**

$C = \{0, -\frac{\sqrt{2}}{5}, \{0, 1\}, \{0\}\}$  \hspace{1cm} |\emph{C}| = 4

no order  \hspace{1cm} = \{\{0\}, 0, \{0, 1\}, -\frac{\sqrt{2}}{5}\}

no repetition  \hspace{1cm} = \{\{0\}, 0, -\frac{\sqrt{2}}{5}, \{0, 1\}, -\frac{\sqrt{2}}{5}\}$

$D = \{0, 1\}$

$E = \{0, \frac{\sqrt{2}}{5}, 0, E\}$

We can describe a set either by listing its elements or by the set-builder technique:

$A = \{a \mid p(a)\}$  \hspace{1cm} \text{all elements } a \text{ such that } p(a)$

$A = \{a \text{ s.t. } p(a)\}$

$A = \{a : p(a)\}$

**Ex**

$N = \{1, 2, 3, 4, 5, \ldots\}$  \hspace{1cm} \text{Set of natural numbers}$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$\mathbb{P} = \{2, 3, 5, 7, 11, 13, \ldots\}$  \hspace{1cm} \text{Set of primes}$
\[ Q = \left\{ \frac{n}{m} : \, n \in \mathbb{Z}, \, m \in \mathbb{N} \text{ and } m \neq 0 \right\} \]

set of real numbers

\[ R \] (is the closure of set of rational numbers, a real number is the limit of a sequence of rational numbers)

1, 1.41, ..., \rightarrow \sqrt{2}

\[ \mathbb{P} = \{ r \in \mathbb{R} : \, r \notin \mathbb{Q} \} \]

Def Real intervals:

\[ [a, b] = \{ r \in \mathbb{R} : \, a \leq r \leq b \} \]

\[ (a, b] = \{ r \in \mathbb{R} : \, a < r \leq b \} \]
\((-\infty, b) = \{ r \in \mathbb{R} : r < b \}\)

notice that \(\infty\) is not a real number.
it is just a notation.

\([a, \infty) = \{ r \in \mathbb{R} : r \geq a \}\)

**Defn** Empty set is the set with no elements denoted by \(\emptyset\) or \(
\{\}\)

\(|\emptyset| = 0\)

**Defn** Given two sets \(A\) and \(B\), we say that set \(A\) is a subset of set \(B\)
and we write \(A \subseteq B\) if every element of \(A\) is an element of \(B\).

We say that set \(A\) is a proper subset of set \(B\) if \(A \subseteq B\) and \(A \neq B\).

we write \(A \subset B\)
equivalently if every element of \(A\) is an element of \(B\) and there are some elements in \(B\) which are not in \(A\).
**Defn** Given set $S$, by power set of $S$ denoted by $\mathcal{P}(S)$ we mean the collection of all subsets of set $S$.

**Remark** \[ |\mathcal{P}(S)| = 2^{|S|} \quad \text{(prove later)} \]

**Ex**
\[ C = \{1, 2, 3\} \]
\[ |C| = 3 \]
\[ \mathcal{P}(C) = \mathcal{P}(\{1, 2, 3\}) \]
\[ |\mathcal{P}(C)| = 2^3 = 8 \]

**Set Operations**

**Defn** (Universal Set) is the background set depending on the context of all elements of interest in our topic, denoted by $U$.

**Defn** Given two arbitrary sets $A$ and $B$
their union is the new set
\[ A \cup B = \{ x : x \in A \text{ or } x \in B \} \]

The intersection of sets $A$ and $B$
is the new set
\[ A \cap B = \{ x : x \in A \text{ and } x \in B \} \]
The difference of sets $A$ and $B$ is defined to be not set $A - B$ also denoted by $A \setminus B$ defined as

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

made of all elements in the universal set $U$ which belongs to $A$ but not to $B$.

All called relative complement of $B$ in $A$.

**Defn**

Given universal set $U$ and set $A \subseteq U$ we define complement of $A$ denoted by $\overline{A}$ or $A^c$ to be the set of all elements of our universal set which are not in $A$.

$$\overline{A} = \{ x \in U : x \notin A \}$$

**Venn diagram** Use a big rectangle for universal set $U$ and use small circles or ellipses to denotes sets, points inside a circle is as set. (not a proof)
$A \cap B$

$A \cup B$

$\overline{A}$

$\overline{\emptyset} = U$

$A \setminus B$

$B \setminus A$
Ex

\[ U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \]
\[ A = \{ 2, 3, 5, 7, 8, 9 \} \]
\[ B = \{ 2, 8 \} \]
\[ C = \{ 4, 6, 7, 10 \} \]

Find \( \overline{C} \), \( B \setminus C \), \( A \cap C \), \( A \cup C \), \( A \cup C \)

\[ \overline{C} = \{ 1, 2, 3, 5, 8, 9 \} \]
\[ B \setminus C = \{ 2, 8 \} = B \]
\[ A \cap C = \{ 7 \} \]
\[ A \cup C = \{ 2, 3, \ldots, 10 \} \]
\[ A \cup C = \{ 1 \} \]
Write all subsets of set $S = \{1, 2\}$

- $S_1 = \emptyset = \{\}$
- $S_2 = \{1\}$
- $S_3 = \{2\}$
- $S_4 = \{1, 2\}$

$2^S = \mathcal{P}(S) = \{S_1, S_2, S_3, S_4\}$

- $|S| = 2$
- $|\mathcal{P}(S)| = 4 = 2^2$

$2^S = \mathcal{P}(\emptyset) = \{\emptyset\}$

- $|\emptyset| = 0$
- $|\mathcal{P}(\emptyset)| = 1 = 2^0$

Let $A = \{1\}$

$\mathcal{P}(A) = \{\emptyset, A\}$

- $|A| = 1$
- $|\mathcal{P}(A)| = 2 = 2^1$

$B = \{\emptyset, \{\emptyset\}\}$

- $B_1 = \emptyset = \{\}$
- $B_2 = \{\emptyset\}$
- $B_4 = \{\emptyset, \{\emptyset\}\} = B$

$\mathcal{P}(B) = \{\emptyset, B_1, B_2, B_3, B_4\}$

- $B_3 = \{\emptyset\}$
Ex Using a Venn diagram, check that

\[ A - B = A \cap \overline{B} \]