1. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on $A$ that is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) reflexive but neither symmetric nor transitive
   (e) symmetric but neither reflexive nor transitive
   (f) transitive but neither reflexive nor symmetric

2. Let $A = \{1\}$ and $B = \{1, 2\}$.
   (a) Determine all relations on $A$.
   (b) Determine all relations from $A$ to $B$.
   (c) Determine all equivalence relations on $B$.

3. Let $A$ be a non-empty set. Let $R = \emptyset$. Is $R$ an equivalence relation on $A$: why or why not?

4. Let $n \geq 2$. Let $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$ denote the integers modulo $n$. Recall that the elements of $\mathbb{Z}_n$ are the distinct equivalence classes induced by the equivalence relation $aRb$ if and only if $a \equiv b \mod n$.
   Show that multiplication in $\mathbb{Z}_n$ is well-defined.
   Note: By Theorem 8.2, it suffices to verify that for all $a, b \in \mathbb{Z}_n$ and any $x \in [a]$ and $y \in [b]$ we have that $xy \in [ab]$.
   Hint: see Result 4.11.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. For $x, y \in \mathbb{R}$, write $xRy$ if and only if $f(x) = y$.
   (a) Verify that $R$ is a relation on $\mathbb{R}$.
   (b) Is $R$ an equivalence relation: why or why not?
   (c) Find the domain and range of $R$, $\text{dom}(R)$ and $\text{rng}(R)$.
   (d) Describe the inverse relation $R^{-1}$.
   (e) Sketch $R$ and $R^{-1}$.
   (f) Does $R^{-1}$ correspond to the graph of a function: why or why not?

6. Let $S = \{2^n : n \in \mathbb{Z}\}$. Define a relation $R$ on $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$ by $qRr$ if and only if $q/r \in S$.
   (a) Show that $R$ is an equivalence relation.

7. Define a relation $R$ on $\mathbb{Z}$ by $aRb$ if and only if $a \equiv b \mod 2$ and $a \equiv b \mod 3$. Prove or disprove: $R$ is an equivalence relation on $\mathbb{Z}$.

8. The relation $R$ on $\mathbb{Z}$ defined by $aRb$ if and only if $a^3 \equiv b^3 \mod 4$ is an equivalence relation (Verify this on your own, but do not hand it in for marks). Determine the distinct equivalence classes.

9. Let $A = \{ a + b\sqrt{2} : a, b \in \mathbb{Q}, a + b\sqrt{2} \neq 0 \}$.
Let $R$ be the relation on $A$ defined by $xRy$ if and only if $x/y \in \mathbb{Q}$. Then $R$ is an equivalence relation (Verify this on your own, but do not hand it in for marks). Determine the distinct equivalence classes.

10. The factorial of a positive integer $n$ is defined by

$$ n! = 1 \cdot 2 \cdot \ldots \cdot n $$

(a) Prove that, for every positive integer $n$, $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n + 1)! - 1$.

(b) Prove that every positive integer $m$ can be written in ‘base factorial’, i.e. in the form

$$ m = a_1 \cdot 1! + a_2 \cdot 2! + \ldots + a_n \cdot n! $$

for some positive integer $n$ and integers $a_1, \ldots, a_n$ which have the property that $0 \leq a_i \leq i - 1$ for each $i \in \{1, \ldots, n\}$. 