5.23 Prove there is no integer \( a \) such that \( a \equiv 5 \pmod{14} \) and \( a \equiv 3 \pmod{21} \).

5.55 Prove that there do not exist positive integers \( a \) and \( n \) such that \( a^2 + 3 = 3^n \).

3. Carefully read the proof (from the class notes or from the book) that \( \sqrt{2} \) is irrational. Then prove the following.
   
   (a) Prove that \( \sqrt[3]{5} \) is irrational.  
   
   (b) Prove that \( \log_2(5) \) is irrational. (Remember that \( \log_a(b) \) is the unique real number such that \( a^{\log_a(b)} = b \).) 

   (c) (Bonus) Prove that if \( n \) is a positive integer such that \( n \geq 2 \), then \( \sqrt[3]{5}/3 \) is irrational.

4. Prove that between any two real numbers, there are infinitely many rational numbers. (Hint: First prove as a lemma that between any two real numbers, there is at least one rational number.)

5. Consider the following proposition.

**Proposition 0.1.** Let \( x \) be any positive real number. Then for every positive real number \( y \), there is a positive real number \( z \) such that \( z^x > y \).

   (a) The following is an invalid proof. Explain why it is invalid.

   \[ \text{Proof.} \text{ Suppose that the statement is false. Then for some } x, \text{ there is a positive real number } y \text{ such that for every } z > 0, z^x \leq y. \text{ Either } y > 1 \text{ or } y \leq 1: \text{ let us consider these two cases separately.} \]

   \[ \begin{itemize}
   \item If } y > 1, \text{ then set } z = y \text{ and } x = 2. \text{ Then } z^x = y^2 > y. \text{ But this contradicts the assumption that } z^x \leq y, \text{ so we have a contradiction!}
   \item If } y \leq 1, \text{ then set } z = 2 \text{ and } x = 1. \text{ Then } z^x = 2 > y. \text{ But this contradicts the assumption that } z^x \leq y, \text{ so we have a contradiction!}
   \end{itemize} \]

   In each case we reach a contradiction, and so our assumption was incorrect. Therefore, for every positive real number \( x \) and every positive real number \( y \), there is a positive real number \( z \) such that \( z^x > y \).

   \( \square \)

   (b) Give a correct proof of the proposition.

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Suppose that \[ \exists a, b \in \mathbb{R}, (a < b \land f(a) < f(b)) \]

and \[ \exists c, d \in \mathbb{R}, (c < d \land f(c) > f(d)) \]

Use the Intermediate Value Theorem to prove that \[ \exists x, y \in \mathbb{R}, (x < y \land f(x) = f(y)) \]

**Hint:** Construct a continuous function \( H(t) \) with the property that \( H(0) = f(b) - f(a) \) and \( H(1) = f(d) - f(c) \). Then use the Intermediate Value Theorem on \( H(t) \) over the interval \([0, 1] \).