5.23 Prove there is no integer $a$ such that $a ≡ 5 \pmod{14}$ and $a ≡ 3 \pmod{21}$.

5.55 Prove that there do not exist positive integers $a$ and $n$ such that $a^2 + 3 = 3^n$.

3. Carefully read the proof (from the class notes or from the book) that $\sqrt{2}$ is irrational. Then prove the following.

(a) Prove that $\sqrt{5}$ is irrational.

(b) Prove that $\log_2(5)$ is irrational. (Remember that $\log_a(b)$ is the unique real number such that $a^{\log_a(b)} = b$.)

(c) (Bonus) Prove that if $n$ is a positive integer such that $n ≥ 2$, then $\sqrt{5}/3$ is irrational.

4. Prove that between any two real numbers, there are infinitely many rational numbers. (Hint: First prove as a lemma that between any two real numbers, there is at least one rational number.)

5. Consider the following proposition.

**Proposition 0.1.** Let $x$ be any positive real number. Then for every positive real number $y$, there is a positive real number $z$ such that $z^x > y$.

(a) The following is an invalid proof. Explain why it is invalid.

*Proof.* Suppose that the statement is false. Then for some $x$, there is a positive real number $y$ such that for every $z > 0$, $z^x ≤ y$. Either $y > 1$ or $y ≤ 1$: let us consider these two cases separately:

- If $y > 1$, then set $z = y$ and $x = 2$. Then $z^x = y^2 > y$. But this contradicts the assumption that $z^x ≤ y$, so we have a contradiction!
- If $y ≤ 1$, then set $z = 2$ and $x = 1$. Then $z^x = 2 > y$. But this contradicts the assumption that $z^x ≤ y$, so we have a contradiction!

In each case we reach a contradiction, and so our assumption was incorrect. Therefore, for every positive real number $x$ and every positive real number $y$, there is a positive real number $z$ such that $z^x > y$. □

(b) Give a correct proof of the proposition.

6. Let $f : \mathbb{R} → \mathbb{R}$ be a continuous function. Suppose that

$$\exists a, b ∈ \mathbb{R}, (a < b \land f(a) < f(b))$$

and

$$\exists c, d ∈ \mathbb{R}, (c < d \land f(c) > f(d))$$

Use the Intermediate Value Theorem to prove that

$$\exists x, y ∈ \mathbb{R}, (x < y \land f(x) = f(y))$$