1. Write down the following
   (a) The completeness axiom
   (b) Definition of max S, sup S of a nonempty subset of \( \mathbb{R} \)
   (c) Definition of \( \lim_{n \to \infty} a_n = L \)
   (d) Definition of a sequence \( \{a_n\} \) is bounded, \( a_n \in \mathbb{R} \).
   (e) Definition of \( \sum_{n=1}^{\infty} a_n \) converges.

2. Consider the telescoping series
   \[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)}. \]
   By calculating the partial sums, prove that the series is convergent. Find its limit.

3. Consider the sequence
   \[ \left\{ a_n = \frac{\sin(n) + 1}{n^2} ; \ n \in \mathbb{N} \right\}. \]
   Use the \( \epsilon - N \) definition for the limit of a sequence to prove that the sequence \( a_n \) converges to zero as \( n \to \infty \).

4. Use two methods to show \( \sum_{n=1}^{\infty} \frac{n^3}{2^n} \) converges.

**Solution:**
- Method 1. By the previous question, for \( n \geq 100 \),
  \[ \frac{n^3}{2^n} \leq \frac{1}{n^2}. \]
  Note that \( \sum_{n=1}^{\infty} 1/n^2 \) converges, so by comparison test, \( \sum_{n=100}^{\infty} n^5/2^n \) converges and adding finitely many (first 99 terms) does not affects convergence, so \( \sum_{n=1}^{\infty} n^5/2^n \) converges.
- Method 2.
  \[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{n^3} \cdot \frac{2^n}{2^{n+1}} = \left( 1 + \frac{1}{n} \right)^3 \cdot \frac{1}{2} \to \frac{1}{2} < 1, \text{ as } n \to \infty. \]
  Ratio test then implies the series is convergent.

5. Prove: If \( \lim_{n \to \infty} a_n = 0 \), then for any \( k > 0 \), \( \lim_{n \to \infty} a_n^k = 0 \).

**Solution:** So suppose that \( k > 0 \) and \( a_n \to 0 \). Choose any \( \epsilon > 0 \). Then there exists an \( N \in \mathbb{R} \) so that \( n > N \) implies that \( |a_n - 0| < \epsilon^{1/k} \). So then \( n > N \) implies that
  \[ |a_n^k - 0| = |a_n|^k < \left( \epsilon^{1/k} \right)^k = \epsilon. \]
  This shows that \( a_n^k \to 0 \).
6. Let \( \{a_n\} \) be a sequence of positive numbers. Show: \( \lim_{n \to \infty} a_n = \infty \) iff \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).

**Solution:** "\( \implies \)". Given any \( \epsilon > 0 \), let \( M = 1/\epsilon \). Then there is a number \( N \) s.t. \( n > N \) implies that \( a_n > M = 1/\epsilon \). Since each \( a_n \) is positive we have: for \( n > N \)

\[
\left| \frac{1}{a_n} - 0 \right| < \epsilon.
\]

Thus \( \lim 1/a_n = 0 \).

"\( \impliedby \)". So suppose that \( a_n \) is a sequence of positive numbers with \( 1/a_n \to 0 \). To show \( a_n \to \infty \), choose any \( M \in \mathbb{R} \). Since \( a_n > 0 \) for each \( n \), then if \( M \leq 0 \) then we automatically have \( a_n > M \) for all \( n \in \mathbb{N} \). So suppose that \( M > 0 \).

Then \( 1/a_n \to 0 \) implies that there is a number \( N \) so that \( n > N \) implies that \( 1/a_n < 1/M \). Hence, \( n > N \) implies that \( M < a_n \), which shows that \( a_n \to \infty \).

7. Let \( S,T \) be sets. Prove the following

(a) If \( |S| \leq |T| \) then \( |\mathcal{P}(S)| \leq |\mathcal{P}(T)| \).

**Solution:**

Proof. Assume \( |S| \leq |T| \) and hence there is an injection from \( f : |S| \to |T| \). We must construct an injection from \( \mathcal{P}(S) \to \mathcal{P}(T) \).

Let \( g : \mathcal{P}(S) \to \mathcal{P}(T) \) be defined by

\[
g(A) = f(A) = \{f(a) | a \in A\}
\]

ie - we just apply \( f \) to each element of \( A \).

Now let \( A,B \in \mathcal{P}(S) \) and assume that \( g(A) = g(B) \). We now show \( A = B \).

- Let \( x \in A \). Hence \( f(x) = y \in f(A) = g(A) \). Since \( g(A) = g(B) \) we must have \( y \in g(B) \). Thus there is some \( z \in B \) so that \( y = f(z) \). But then since \( f \) is injective, \( x = z \) and so \( x \in B \). Hence \( A \subseteq B \).

- The reverse inclusion is similar. Let \( z \in B \). Hence \( f(z) = y \in f(B) = g(B) \). Since \( g(A) = g(B) \) we must have \( y \in g(A) \). Thus there is some \( x \in A \) so that \( y = f(x) \). But then since \( f \) is injective, \( x = z \) and so \( z \in A \). Hence \( B \subseteq A \).

Thus \( g \) is injective and the result follows.

(b) If \( |S| = |T| \) then \( |\mathcal{P}(S)| = |\mathcal{P}(T)| \).

**Solution:**

Proof. Assume \( |S| = |T| \). Hence there is a bijection \( f : S \to T \). We must construct an injection from \( \mathcal{P}(S) \to \mathcal{P}(T) \). We use the same function as above.

Let \( g : \mathcal{P}(S) \to \mathcal{P}(T) \) be defined by

\[
g(A) = f(A) = \{f(a) | a \in A\}
\]

where \( A \in \mathcal{P}(S) \)}
By the previous question we know that \( g \) is injective. It suffices to prove that \( g \) is also surjective. Let \( B \in \mathcal{P}(T) \) and since \( f \) is a bijection, its inverse exists and we may set

\[
A = \{ f^{-1}(b) | b \in B \}
\]

We must now prove that \( g(A) = B \).

- Let \( x \in g(A) \). Then \( x = f(a) \) for some \( a \in A \). But then \( a = f^{-1}(b) \) for some \( b \in B \). Hence \( x = f(f^{-1}(b)) = b \). So \( x \in B \).
- Now let \( x \in B \). By construction \( f^{-1}(x) \in A \). Hence \( f(f^{-1}(x)) \in f(A) = g(A) \).

Thus \( g \) is surjective and we are done.

8. Consider the function \( f : (-a,a) \to \mathbb{R} \) defined by \( f(x) = \frac{x}{a^2-x^2} \) where \( a > 0 \) is a fixed number and \( x \in (-a,a) \).

(i) Show \( f \) is bijective.

(ii) What can you conclude about the cardinalities of \((-a,a)\) and \(\mathbb{R}\)?

(iii) What can you conclude about the cardinalities of \((-a,a)\) and \((-b,b)\) for \(a,b > 0\)?

**Solution:**

(i) **Surjectivity:** For any \( y \in \mathbb{R} \), we show there is \( x \in (-a,a) \) s.t. \( f(x) = y \). If \( y = 0 \), take \( x = 0 \in (-a,a) \), then \( f(0) = 0 \). If \( y \neq 0 \), we solve \( y = x/(a^2 - x^2) \) which implies \( yx^2 + x - a^2y = 0 \), and hence

\[
x_\pm = \frac{-1 \pm \sqrt{1 + 4a^2y^2}}{2y}.
\]

We need to check which of the two solutions (if any) lies in \((-a,a)\). We claim \( x_+ \in (-a,a) \).

- If \( y > 0 \), then \( 0 < -1 + \sqrt{1 + 4a^2y^2} < 2ay \) because \( 1 < \sqrt{1 + 4a^2y^2} \) (showing the 1st "<") and \( 1 + 4a^2y^2 < 1 + 4a^2y^2 + 4ay = (1 + 2ay)^2 \) (showing the 2nd "<" by taking square root). So we conclude that if \( y > 0 \) then \( x_+ \in (0,a) \).
- If \( y < 0 \), then \( 0 < -1 + \sqrt{1 + 4a^2y^2} < -2ay \), because \( 1 < \sqrt{1 + 4a^2y^2} \) (for 1st "<") and \( 1 + 4a^2y^2 < 1 + 4a^2y^2 - 4ay = (1-2ay)^2 \) (for 2nd "<" by taking square root). We conclude \( x_+ \in (-a,0) \). **Careful:** dividing the negative number \( 2y \) changes direction of inequality!

This shows \( f \) is surjective.

Note

\[
|x_+| = \frac{|1 + \sqrt{1 + 4a^2y^2}|}{2|y|} \geq \frac{\sqrt{1 + 4a^2y^2}}{2|y|} \geq \frac{4a^2y^2}{2|y|} = |a|.
\]

Thus \( x_+ \notin (-a,a) \). So we **CANNOT** use \( x_- \) for surjectivity of \( f \).
Injectivity: If \( f(x_1) = f(x_2) \), then \( x_1/(a^2 - x_1^2) = x_2/(a^2 - x_2^2) \). This implies \( x_1(a^2 - x_2^2) - x_2(a^2 - x_1^2) = 0 \). Then \( (x_1 - x_2)(a^2 + x_1x_2) = 0 \). Since \(|x_1| < a \) and \(|x_2| < a \), we have \(|x_1x_2| < a^2 \), therefore \( a^2 + x_1x_2 \neq 0 \). So \( x_1 - x_2 = 0 \), and \( f \) is injective.

(ii) \(|(-a, a)| = |\mathbb{R}|\).

(iii) Write \( f_a : (-a, a) \to \mathbb{R} \) defined by \( f_a(x) = x/(a^2 - x^2) \) and \( f_b : (-b, b) \to \mathbb{R} \) given by \( f_b(x) = x/(b^2 - x^2) \). By (i), \( f_a, f_b \) are both bijective. Hence \( f_b^{-1} \circ f_a : (-a, a) \to (-b, b) \) is bijective. It follows \(|(-a, a)| = |(-b, b)|\).

9. Show the following pairs of sets \( S \) and \( T \) are equinumerous by finding a specific bijection between the sets in each pair (you need to prove your function is bijective).

(a) \( S = [0, 1] \) and \( T = [1, 4] \)

Solution:

\[ f(x) = 1 + 3x \]

\[ 1 + 3x = 1 + 3z \]

\[ \text{Hence } f \text{ is injective.} \]

• Surjective: Let \( y \in [1, 4] \) and set \( x = \frac{y - 1}{3} \). Since \( y \geq 1 \), \( x > 0 \) and since \( y \leq 4 \), \( x \leq 1 \). Further

\[ f(x) = 1 + 3 \cdot \frac{y - 1}{3} = y \]

as required. Hence \( f \) is surjective.

(b) \( S = (0, 1) \) and \( T = (0, \infty) \)

Solution:

\[ g(x) = \frac{1}{x} - 1 \]

\[ \frac{1}{x} - 1 = \frac{1}{z} - 1 \]

\[ \frac{1}{x} = \frac{1}{z} \]

\[ z = x \]

Thus \( g \) is injective.
• Surjective: Let \( y \in (0, \infty) \) and then set \( x = \frac{1}{y+1} \). Since \( y \in (0, \infty) \) we must have \( 0 < x < 1 \). Then

\[
g(x) = \frac{1}{y+1} - 1
= \frac{y + 1}{1} - 1 = y
\]

Hence \( g \) is surjective.

(c) \( S = [0, 1] \) and \( T = [0, 1) \)

• Hint for (c) — define your function using 2 cases: \( x = 1/n \) and \( x \neq 1/n \) for \( n \in \mathbb{N} \).

Then consider why \( |\{1, 1/2, 1/3, 1/4, \ldots\}| = |\{1/2, 1/3, 1/4, \ldots\}| \)

Solution: The key here is to take \( x = \frac{1}{n} \) and map it to \( \frac{1}{1+1/x} = \frac{1}{1+\frac{1}{a}} \). This “shuffles” \( 1 \mapsto \frac{1}{2} \) and \( \frac{1}{2} \mapsto \frac{1}{3} \) and so on.

Proof. Let \( f : S \to T \) be defined by

\[
f(x) = \begin{cases} \frac{1}{1+\frac{1}{x}} & \text{if } x = 1/n, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}
\]

We prove that \( f \) is a bijection.

• Injective: Let \( x, z \in S = [0, 1] \) and assume \( x \neq z \). There are four cases to consider depending on whether or not \( \frac{1}{x}, \frac{1}{z} \in \mathbb{N} \) or not.

  – If \( \frac{1}{x} = a, \frac{1}{z} = b \) with \( a, b \in \mathbb{N} \) then we must have \( a \neq b \). Hence \( 1 + a \neq 1 + b \) and so \( \frac{1}{1+a} = f(x) \neq f(z) = \frac{1}{1+b} \).

  – If \( \frac{1}{x}, \frac{1}{z} \notin \mathbb{Z} \) then \( f(x) = x \) and \( f(z) = z \) and so \( f(x) \neq f(z) \).

  – If \( \frac{1}{x} = a \in \mathbb{N} \) and \( \frac{1}{z} \notin \mathbb{N} \) then we have \( f(x) = \frac{1}{1+a} \) and \( f(z) = z \). Assume, to the contrary that \( f(x) = f(z) \). Then \( \frac{1}{x} = 1 + a \in \mathbb{N} \) — this gives a contradiction. Hence \( f(x) \neq f(z) \).

  – If \( \frac{1}{x} = a \notin \mathbb{N} \) and \( \frac{1}{z} \in \mathbb{N} \) then we have \( f(z) = \frac{1}{1+a} \) and \( f(x) = x \). Assume, to the contrary that \( f(x) = f(z) \). Then \( \frac{1}{x} = 1 + a \in \mathbb{N} \) — this gives a contradiction. Hence \( f(x) \neq f(z) \).

In all four cases, if \( x \neq z \) then \( f(x) \neq f(z) \), and so \( f \) is an injection.

• Surjective: Let \( y \in [0, 1) \). Then there are 2 cases to consider. If \( y = 1/n \) for \( n \in \{2, 3, \ldots\} \) or not.

  – If \( y = 1/n \) then set \( x = \frac{1}{y-1} \). Since \( y = 1/n, x = 1/(n-1) \) and so \( x \) is the reciprocal of a natural number and \( 0 < x < 1 \). Hence

\[
f(x) = \frac{1}{1+\frac{1}{y-1}} = y
\]
If \( y \neq 1/n \) for any \( n \in \mathbb{N} \) then set \( x = y \). Now \( x \in [0, 1] \) and \( f(x) = y \).

Hence \( f \) is surjective.

10. Using the guide given below, complete the proof of the following very useful criteria for determining when a set is countable.

**Theorem.** Let \( S \) be a nonempty set. The following three conditions are equivalent.

(a) \( S \) is countable.

(b) There exists an injection \( f : S \to \mathbb{N} \).

(c) There exists a surjection \( g : \mathbb{N} \to S \).

Guide — split the proof into 3 implications as indicated below.

(1) First prove (a) \( \implies \) (b): \( S \) is countable implies there is a bijection \( h \) from \( I_n = \{1, \ldots, n\} \) or from \( \mathbb{N} \) to \( S \). Consider its inverse.

(2) Second prove (b) \( \implies \) (c): hint - \( f \) is a bijection from \( S \) to \( f(S) \), hence \( f^{-1} : f(S) \to S \) is defined. Define \( g : \mathbb{N} \to S \) by \( g(n) = f^{-1}(n) \) if \( n \in f(S) \) and \( g(n) = s_0 \) if \( n \notin f(S) \) for some fixed \( s_0 \in S \).

(3) Finally prove (c) \( \implies \) (a): For the given surjective \( g : \mathbb{N} \to S \), define \( h : S \to \mathbb{N} \) by \( h(n) \) the smallest \( i \in \mathbb{N} \) s.t. \( g(n) = s_i \). Then \( h(S) \) is countable (you can use the fact that any subset of a countable set is countable). Finally, draw conclusion on \( S \).

**Solution:** We split the proof into the 3 parts as suggested above.

Assume (a) is true. Hence \( S \) is either finite or denumerable.

- If \( S \) is denumerable, then by definition there is a bijection \( f : S \to \mathbb{N} \) and we are done.

- If \( S \) is finite then there must be a bijection \( h : \{1, 2, \ldots, n\} \to S \) for some \( n \in \mathbb{N} \). The inverse \( h^{-1} \) defines a bijection from \( \mathbb{N} \to \{1, 2, \ldots, n\} \). Define \( g : \{1, 2, \ldots, n\} \to \mathbb{N} \) by \( g(i) = i \), and \( h \) is injective. Then \( g \circ h^{-1} : S \to \mathbb{N} \) is injective as it is the composition of two injections (by a theorem from class).

Assume (b) is true, so that there is an injection \( f : S \to \mathbb{N} \). Define \( T = f(S) \subseteq \mathbb{N} \), let \( h : S \to T \) be a function defined by \( h(x) = f(x) \). Then \( h \) is injective (since \( f \) is injective) and also surjective. To see this, let \( t \in T = f(S) \). By definition of the image of a set, there is \( x \in S \) so that \( t = f(x) = h(x) \).

Now, since \( h \) is bijective, its inverse \( h^{-1} : T \to S \) exists. Now let \( q \in S \), and define \( g : \mathbb{N} \to S \) by

\[
g(n) = \begin{cases} h^{-1}(n) & \text{if } n \in f(S) = T \\ q & \text{otherwise} \end{cases}
\]

We claim that this function \( g \) is a surjection. Let \( y \in S \) and set \( z = f(y) \in f(S) \). Then \( g(z) = h^{-1}(z) = f^{-1}(z) = y \) (since \( f \) is a bijection from \( S \) to \( f(S) \), this inverse is well defined). Hence \( g \) is a surjection.
Assume (c) is true. Hence there is a surjection \( g : \mathbb{N} \to S \). Let \( y \in S \). Since \( g \) is surjective, the set

\[
g^{-1}(\{y\}) = \{n \in \mathbb{N} : g(n) = y \} \subseteq \mathbb{N}
\]

is non-empty. Define a function \( h : S \to \mathbb{N} \) by

\[
h(y) = \text{the minimum element of } g^{-1}(\{y\}).
\]

We claim \( h \) injective: If \( h(y) = h(w) \) which equals some \( k \in \mathbb{N} \), then \( k = \text{the minimum element of } g^{-1}(\{y\}) \) and \( k = \text{the minimum element of } g^{-1}(\{w\}) \). This means \( g(k) = y \) and \( g(k) = w \), hence \( y = w \) as \( g \) is a function. So \( h \) is an injection from \( S \to \mathbb{N} \) and thus is a bijection from \( S \to h(S) \subseteq \mathbb{N} \). Any subset of \( \mathbb{N} \) must be countable (by a theorem from class) and so \( h(S) \) is countable and thus \( S \) is countable.

11. Prove that \(|(0, \infty)| = |\mathbb{R}| \) by finding a bijection between the two sets.

Hint: look at Theorem 10.13 — it will give you ideas.

**Solution:** Use \( f : (0, \infty) \to \mathbb{R} \) defined by \( f(x) = x - 1/x \).

- **Injective.** Let \( a, b \in (0, \infty) \) and assume \( f(a) = f(b) \). Then
  \[
  a - 1/a = b - 1/b \\
  a^2b - b = ab^2 - a \\
  a^2b - ab^2 + a - b = 0 \\
  (a - b)(ab + 1) = 0
  \]

  Thus we either have \( a = b \) or \( ab = -1 \). Since \( a, b > 0 \) the second cannot happen and so \( a = b \). Thus \( f \) is injective.

- **Surjective — (use quadratic formula to get this)** Let \( y \in \mathbb{R} \) and then let

  \[
  x = \frac{y + \sqrt{y^2 + 4}}{2}
  \]

  Then \( f(x) \) is

  \[
  x - 1/x = \frac{y + \sqrt{y^2 + 4}}{2} - \frac{2}{y + \sqrt{y^2 + 4}} \quad \text{common denominator}
  \]

  \[
  = \frac{(y + \sqrt{y^2 + 4})^2 - 4}{2(y + \sqrt{y^2 + 4})}
  \]

  \[
  = \frac{y^2 + 2y\sqrt{y^2 + 4} + y^2 + 4 - 4}{2(y + \sqrt{y^2 + 4})}
  \]

  \[
  = \frac{2y^2 + 2y\sqrt{y^2 + 4}}{2(y + \sqrt{y^2 + 4})} = y
  \]

  as required. It remains to show that \( x \in (0, \infty) \).
Note that since \( y \in \mathbb{R}, y^2 + 4 > y^2 > 0. \) Hence if \( y > 0, y + \sqrt{y^2 + 4} > 0. \) And if \( y < 0 \) then \( y + \sqrt{y^2 + 4} > 0. \) Thus \( x > 0 \) as required.

Hence \( f \) is surjective.

- Since \( f \) is injective and surjective, it is bijective and the two sets have the same cardinality.

12. Let \( A, B, C, D \) be non-empty sets. Prove that

\[
\text{if } |A| = |C| \text{ and } |B| = |D| \text{ then } |A \times B| = |C \times D|.
\]

Remember — the sets may or may not be finite. This also applies to the remaining questions below.

**Solution:**

- Since \( |A| = |C| \) and \( |B| = |D| \) there exist bijections \( f : A \to C \) and \( g : C \to D. \)
- Define \( h : A \times B \to C \times D \) by \( h(a, b) = (f(a), g(b)). \) We must show it is a bijection.
- Injection. Let \( (a_1, b_1), (a_2, b_2) \in A \times B. \) Assume that \( h(a_1, b_1) = h(a_2, b_2). \) By our definition of \( h \) we know that \( f(a_1) = f(a_2) \) and so \( a_1 = a_2. \) Similarly, \( g(b_1) = g(b_2) \) and so \( b_1 = b_2. \) Thus \( (a_1, b_1) = (a_2, b_2) \) and so \( h \) is injective.
- Surjection. Let \( (c, d) \in C \times D. \) Since \( f \) and \( g \) are surjective there are \( a \in A \) and \( b \in B \) so that \( f(a) = c \) and \( g(b) = d. \) Now \( (a, b) \in A \times B \) and \( h(a, b) = (c, d). \) Thus \( h \) is surjective.
- Since \( h \) is injective and surjective, it is bijective and the two sets have the same cardinality.

13. Let \( A \) be an non-empty set.

(a) Prove that \( |A| \leq |A \times A|. \)

**Solution:**

- It suffices to find an injection from \( A \) to \( A \times A. \)
- Define \( f : A \to A \times A \) by \( f(a) = (a, a). \)
- Let \( a, b \in A \) and assume \( f(a) = f(b). \) Thus \( (a, a) = (b, b) \) and so we must have \( a = b. \) Hence \( f \) is injective.

(b) Prove that \( |A| \leq |\mathcal{P}(A)|. \)

**Solution:**

- It suffices to find an injection from \( A \) to \( \mathcal{P}(A) \) (we did this in a previous homework).
14. Prove the following theorem:

**Theorem.** Let $A, B, C$ be nonempty sets. Then

(a) If $A \subseteq B$ then $|A| \leq |B|$.
(b) $|A| \leq |A|$.
(c) If $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$.

**Solution:**

- Define $f : A \to B$ by $f(a) = a$. It is clearly an injection.
- Define $f : A \to A$ by $f(a) = a$. This is the identity function which is bijective and so injective.
- Assume $|A| \leq |B|$ and $|B| \leq |C|$. Thus there are injections $f : A \to B$ and $g : B \to C$. By theorem 9.7 the compositions of injections is also an injection, so $g \circ f : A \to C$ is an injection.

15. Let $A, B$ be sets. Prove that

if $|A - B| = |B - A|$ then $|A| = |B|$.

Hint: draw a careful picture

Hint2: Given a bijection $f : (A - B) \to (B - A)$ define

$$g : A \to B$$

$$g(x) = \begin{cases} f(x) & x \in (A - B) \\ x & x \notin (A - B) \end{cases}$$

**Solution:**

![Diagram showing set operations and bijection](image)
• Let \( g : A \to B \) be defined as above. We need to show that \( g \) is injective and surjective.

• Injective. Let \( x, z \in A \) and assume \( g(x) = g(z) \). This image must be in \( B \), but it may either be in \( A \) or not in \( A \) (that is, either \( y \in A \cap B \) or \( y \in B - A \)).
  
  – Assume \( g(x) = g(z) \notin A \). Then both \( x, z \in A - B \) (otherwise their images under \( g \) would be in \( A \)). Hence \( g(x) = f(x) \) and \( g(z) = f(z) \). Since \( f \) is injective, it follows that \( x = z \).
  
  – Now assume that \( g(x) = g(z) \in A \). Then both \( x, z \in A \) (otherwise their images under \( g \) would be in \( B - A \)). Then \( g(x) = x \) and \( g(z) = z \) and so \( x = z \).

Hence \( g \) is injective.

• Surjective. Let \( y \in B \). Either \( y \in A \) or \( y \notin A \) (that is, either \( y \in A \cap B \) or \( y \in B - A \)).
  
  – Assume \( y \in A \) then let \( x = y \). By the definition of \( g \), \( g(x) = x = y \).
  
  – Now assume \( y \notin A \), then since \( f \) is surjective, there exists \( x \in A - B \) so that \( f(x) = y \). Now since \( x \in A - B \), it follows that \( g(x) = f(x) = y \).

Hence \( g \) is surjective.

• Hence \( g(x) \) is bijective as required.