1. (a) Write the negation of the following statement:

“There is some \((a, b) \in \mathbb{N} \times \mathbb{N}\) so that \(a \leq b\) and \(a^2 \leq b^2\).”

**Solution:** \(\forall (a, b) \in \mathbb{N} \times \mathbb{N}, (a > b)\) or \((a^2 > b^2)\).

(b) Write the negation of the following statement:

“The integer \(n\) is even if and only if \(n^2 + 1\) is even.”

**Solution:** “The integer \(n\) is even if and only if \(n^2 + 1\) is odd.”

(c) Write the converse and contrapositive of the following statement:

“If \(n\) is even, then \(n + 1\) is odd.”

**Solution:** Converse — “If \(n + 1\) is odd, then \(n\) is even.”
Contrapositive — “If \(n + 1\) is even, then \(n\) is odd.”

(d) Give a precise definition of a *set partition*.

**Solution:**

**Definition.** Let \(A\) be a set. A partition of \(A\) is a collection \(S\) of subsets of \(A\) such that

- every \(X \in S\) is non-empty
- for every \(X, Y \in S\), either \(X = Y\) or \(X \cap Y = \emptyset\)
- \(\bigcup_{X \in S} X = A\).

2. (a) State the principle of mathematical induction.

**Solution:** Given statements \(P(n)\) defined for naturals \(n\):

\[ P(1) \land \left( \forall n \in \mathbb{N};\; P(n) \Rightarrow P(n + 1) \right) \Rightarrow (\forall n \in \mathbb{N};\; P(n)) \]

(b) State the well ordering principle of natural numbers.

**Solution:** Every non-empty subset of natural numbers has a minimum.

(c) Prove that \(\sqrt{5}\) is an irrational number.

**Solution:**

*Proof.* Assume, to the contrary, that \(\sqrt{5} \in \mathbb{Q}\).
Hence we can write $\sqrt{5} = \frac{a}{b}$ with $a, b \in \mathbb{N}$. If $a, b$ have a common factor except one we can always eliminate it and so we can assume that $a, b$ have no common factors larger than one.

Now $5 = \frac{a^2}{b^2}$ and so $a^2 = 5b^2$. Thus $a^2$ is divisible by 5, this shows that $a$ is divisible by 5. (Because if $a$ is not divisible by 5 then $a \equiv r \pmod{5}$ with some remainder $r \in \{1, 2, 3, 4\}$. This implies that $a^2 \equiv r^2 \pmod{5}$ with some remainder $s \in \{1, 4\}$. So $a^2$ is not divisible by 5.)

So we can write $a = 5k$ for some $k \in \mathbb{Z}$. Then $5b^2 = 25k^2$ and so $b^2 = 5k^2$. So $b^2$ is divisible by 5. Therefore $b$ is divisible by 5.

Since $a$ and $b$ are both divisible by 5, they have a common factor of 5. This contradicts our assumption and so $\sqrt{5} \not\in \mathbb{Q}$. \( \square \)

(d) What is the remainder of the division, $19^8 - 65^5$ divided by 9.

**Solution:** We have that $19 \equiv 1 \pmod{9}$ and $65 \equiv 2 \pmod{9}$, therefore $19^8 \equiv 1^8 \pmod{9}$ and $65^5 \equiv 2^5 \pmod{9}$. These imply that $19^8 - 65^5 \equiv 1 - 32 \equiv 5 \pmod{9}$. So the remainder of the division, $19^8 - 65^5$ divided by 9 is equal to five.

3. Determine whether the following six statements are true or false — explain your answers (“true” or “false” is not sufficient).

(i) $\forall x \in S, \forall y \in S, xy = 3$.

(ii) $\forall x \in S, \exists y \in S \text{ s.t. } xy = 3$.

(iii) $\exists x \in S \text{ s.t. } \forall y \in S, xy = 3$.

(iv) $\exists x \in S \text{ s.t. } \exists y \in S \text{ s.t. } xy = 3$.

(v) $\exists x \in S \text{ s.t. } \forall y \in S, \forall z \in S, \text{ if } z > y, \text{ then } z \geq x + y$.

(vi) $\forall x \in S, \exists y \in S \text{ s.t. } \forall z \in S, \text{ if } z > y, \text{ then } z \geq x + y$.

**Solution:**

(i) False — the negation is $\exists x, \exists y \text{ s.t. } xy \neq 3$. Pick $x = 1, y = 1$ then $xy = 1 \neq 3$. Since the negation is true, the original must be false.

(ii) True — for any $x$, pick $y = 3/x$. Then no matter what choice of $x$, $xy = 3$.

(iii) False — the negation is $\forall x, \exists y \text{ s.t. } xy \neq 3$. For any given $x$, pick $y = 1/x$, then $xy = 1 \neq 3$. Since the negation is true, the original must be false.

(iv) True — take $x = 1, y = 3$. 

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4. (a) Let \( n \in \mathbb{Z} \). Prove that \( 7n + 1 \) and \( 3n - 6 \) have opposite parity.
(That is, they cannot both be odd and they cannot both be even).

**Solution:** The number \( n \) is either even or odd and so we consider the two cases.

- Assume \( n \) is even, so \( n = 2k \) for some \( k \in \mathbb{Z} \). Then \( 7n + 1 = 14k + 1 = 2(7k) + 1 \) and thus odd. On the other hand \( 3n - 6 = 6k - 6 = 2(3k - 3) \) and thus even. Hence \( 7n + 1 \) and \( 3n - 6 \) have opposite parity.

- Now assume that \( n \) is odd, so \( n = 2l + 1 \) for some \( l \in \mathbb{Z} \). Then \( 7n + 1 = 14l + 8 = 2(7l + 4) \) and thus even. On the other hand \( 3n - 6 = 6l - 3 = 2(3l - 2) + 1 \) and thus odd. Hence \( 7n + 1 \) and \( 3n - 6 \) have opposite parity.

(b) Prove that \( a^3 + 2a \equiv 0 \pmod{3} \).

**Solution:**

Proof. We proceed by direct proof. We consider the three possible cases:

- If \( a \equiv 0 \pmod{3} \), then \( a^3 + 2a \equiv 0^3 + (2)(0) \pmod{3} \).

- If \( a \equiv 1 \pmod{3} \), then \( a^3 + 2a \equiv 1^3 + (2)(1) \pmod{3} \). Then since \( 3 \equiv 0 \pmod{3} \), \( a^3 + 2a \equiv 0 \pmod{3} \).

- If \( a \equiv 2 \pmod{3} \), then \( a^3 + 2a \equiv 2^3 + (2)(2) \pmod{3} \). Then since \( 12 \equiv 0 \pmod{3} \), again \( a^3 + 2a \equiv 0 \pmod{3} \).

(c) Let \( n \in \mathbb{Z} \). Prove that if \( 4 \mid (n^2 + 3) \) then \( 2 \mid (n^3 + 2n^2 + 1) \).

**Solution:**

Proof. We first claim that if \( 4 \mid (n^2 + 3) \) then \( n \) is odd. To prove this claim, we use the contrapositive. Assume \( n \) is even, so \( n = 2k \) for some \( k \in \mathbb{Z} \). Then \( n^2 + 3 = 4k^2 + 3 = 2(2k^2 + 1) + 1 \). Since \( 2k^2 + 1 \in \mathbb{Z} \), \( n^2 + 3 \) is odd, and hence is not divisible by 2 and certainly also not divisible by 4. Now, assume \( 4 \mid (n^2 + 3) \). By our claim, \( n \) is odd. By the result in the first part of this question above, \( 2 \mid (n^3 + 2n^2 + 1) \), as required.

5. Let \( A, B, C \) be sets.
(a) Prove that $A \cap B \subseteq A \cup B$.

Solution:
Proof. Let $x \in A \cap B$, and so $x \notin A \cap B$.

- This implies (by de Morgan’s laws) that $x \notin A$ or $x \notin B$.
- Hence $x \in \overline{A}$ or $x \in \overline{B}$.
- And so $x \in \overline{A} \cup \overline{B}$.

(b) Prove that $(A - B) \cup (A - C) \subseteq (A - (B \cap C))$

Solution:
Proof. Assume $x \in (A - B) \cup (A - C)$.

- Hence $x \in (A - B)$ or $x \in (A - C)$.
  - If $x \in (A - B)$ then $x \in A$ and $x \notin B$. Since $x \notin B$, $x \notin B \cap C$.
  - If $x \in (A - C)$ then $x \in A$ and $x \notin C$. Since $x \notin C$, $x \notin B \cap C$.
- In either case, $x \in A$ and $x \notin B \cap C$, and thus $x \in A - (B \cap C)$.
- Hence $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

(c) Prove that $(A \cap B) \cup (A \cap C) \subseteq (A \cap (B \cup C))$.

Solution:
Proof. Let $x \in (A \cap B) \cup (A \cap C)$.

- So $x \in A \cap B$ or $x \in A \cap B$.
  - If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus $x \in B \cup C$ and so $x \in A \cap (B \cup C)$.
  - Similarly, if $x \in A \cap C$, then $x \in A$ and $x \in C$. Thus $x \in B \cup C$ and so $x \in A \cap (B \cup C)$.
- In either case $x \in A \cap (B \cup C)$.
- Hence $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

6. Use mathematical induction to prove the following.
(a) Prove that
\[
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}
\]
is true for all \( n \in \mathbb{N} \).

**Solution:** Let the statement \( P(n) \) to be \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \).

- \( P(1) : \frac{1}{2} = 1 - \frac{1}{2} \). Hence \( P(1) \) is true.
- Assume \( P(k) \) is true for some natural \( k \):
  \[
  \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}
  \]
  We need to show that \( P(k+1) \) is also true:
  \[
  \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}
  \]
  \[
  = 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!}
  \]
  \[
  = 1 - \frac{1}{(k+2)!}
  \]
  Hence \( P(k) \implies P(k+1) \) is true for all natural \( k \).
- By mathematical induction the statement \( P(n) \) is true for all \( n \in \mathbb{N} \).

(b) Let \( a, b \in \mathbb{Z} \). Prove that \( a^n - b^n \) is a multiple of \( a - b \) for all non-negative integers \( n \).

**Solution:**
- When \( n = 0 \), \( a^0 - b^0 = 1 - 1 = 0 \), and zero is a multiple of any integer. Hence \( P(0) \) is true.
- Assume that \( a^n - b^n \) is a multiple of \( a - b \) for some non-negative integer \( n \). We need to show that \( a^{n+1} - b^{n+1} \) is also a multiple of \( a - b \). We know that \( a^n - b^n = q(a - b) \) for some integer \( q \), so \( a^n = b^n + q(a - b) \). Multiply both sides by \( a \), we get: \( a^{n+1} = a \times b^n + q(a - b) \) therefore \( a^{n+1} - b^{n+1} = a \times b^n + q(a - b) - b^{n+1} \). This implies that \( a^{n+1} - b^{n+1} = (a - b)b^n + qa(a - b) = (a - b)(b^n + qa) \). Therefore \( a^{n+1} - b^{n+1} \) is also a multiple of \( a - b \).
- By mathematical induction, the statement is true for all non-negative integers.

(c) Prove that \( \frac{1}{3}n^3 < \sum_{j=0}^{n}(j + 1)(j + 2) \) for all non-negative integers \( n \).
Solution:

- Since \(0 < 2\), \(P(0)\) is true.
- Assume \(\frac{1}{3}k^3 < \sum_{j=0}^{k}(j+1)(j+2)\), for some non-negative integers \(k\). Then we need to show that \(\frac{1}{3}(k+1)^3 < \sum_{j=0}^{k+1}(j+1)(j+2)\). We have

\[
\sum_{j=0}^{k+1}(j+1)(j+2) = \sum_{j=0}^{k}(j+1)(j+2) + (k+2)(k+3)
\]

\[
> \frac{1}{3}k^3 + (k+2)(k+3)
\]

\[
= \frac{1}{3}k^3 + k^2 + 5k + 6
\]

\[
= \frac{k^3 + 3k^2 + 15k + 18}{3} \geq \frac{k^3 + 3k^2 + 3k + 1}{3} = \frac{(k+1)^3}{3}
\]

Thus \(P(k) \Rightarrow P(k+1)\).
- By mathematical induction, the statement is true for all \(n \in \mathbb{N}\).

7. Let \(S\) be a nonempty subset of \(\mathbb{Z} \times \mathbb{Z}\). We say that \(S\) is circular when for all \(a, b, c \in \mathbb{Z}\), if \((a, b) \in S\) and \((b, c) \in S\) then \((c, a) \in S\).

(a) Consider \(S = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } a - b = 3k \text{ for some } k \in \mathbb{Z}\}\).

Prove that \(S\) is circular.

(b) Now consider \(S = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } a, b \text{ have a common factor } q > 1\}\).

Prove that \(S\) is not circular.

Solution:

Proof. Let \(a, b, c \in \mathbb{Z}\).

(a) Relation \(S\) is circular — Assume that \((a, b), (b, c) \in S\). Then \(a - b = 3k\) and \(b - c = 3\ell\) for some \(k, \ell \in \mathbb{Z}\). Now \(c - a = -(c - b) - (a - b) = -3\ell - 3k = 3(-k - \ell)\). Since \(-k - \ell \in \mathbb{Z}\), it follows that \((c, a) \in S\).

(b) Relation \(S\) is not circular — Let \(a = 2, b = 6, c = 3\). Then we see that \((a, b) \in S\) since \(a, b\) have a common factor of 2. And \((b, c) \in S\) since \((b, c)\) have a common factor of 3. But \((3, 2) \notin S\) since the only common factor is 1.

8. Let \(A, B, C\) be sets.

(a) Prove that if \(A \subseteq B\) then \(A - (B \cap C) \subseteq (A - C)\)

Solution:
Proof. Assume \( A \subseteq B \), and let \( x \in A \setminus (B \cap C) \). Hence \( x \in A \) and \( x \notin B \cap C \). Thus \( x \notin B \) or \( x \notin C \). Since \( x \in A \), our assumption implies that \( x \in B \), and so it must be the case that \( x \notin C \). Hence \( x \in A - C \). \( \square \)

(b) Disprove that \( (A - B) - (B - C) = (A - B) - C \).

Solution: Let \( A = C = \{1\} \) and \( B = \emptyset \). Then

- \( A - B = A \) and \( B - C = \emptyset \). Hence \( LHS = A = \{1\} \).
- Now \( A - B = A = \{1\} \) and \( C = \{1\} \), so \( RHS = \emptyset \).

Hence \( LHS \neq RHS \).

9. Decide whether the following are true or false. Prove your answers

(a) Let \( A, B, C \) be sets. Then

\[
(A \times C) - (B \times C) = (A - B) \times C.
\]

Solution:

Proof. We must show both that \( LHS \subseteq RHS \) and \( RHS \subseteq LHS \).

- Let \((x, y) \in (A \times C) - (B \times C)\). Hence \( x \in A \) and \( y \in C \). Additionally \((x, y) \notin (B \times C)\), and thus either \( x \notin B \) or \( y \notin C \). Since we already know that \( y \in C \), it must be the case that \( x \notin B \). Hence \( x \in (A - B) \) and thus \((x, y) \in (A - B) \times C\).

- Now let \((x, y) \in (A - B) \times C\). Thus \( y \in C \) and \( x \in A \) and \( x \notin B \). Thus we must have \((x, y) \in A \times C\), and since \( x \notin B \), it follows that \((x, y) \notin B \times C\). Hence \((x, y) \in (A \times C) - (B \times C)\).

\( \square \)

(b) Let \( n \in \mathbb{N} \), then \( 3^n > n^3 \).

Solution: This is false. Consider \( n = 3 \), then \( LHS = 3^3 = 27 \) and \( RHS = 3^3 = 27 \). But \( 27 \neq 27 \).

10. Let \( A, B, \) and \( C \) be sets. Prove or disprove the following:

- \( A \cup (B - C) = (A \cup B) - (A \cup C) \)
- \( A \cap (B - C) = (A \cap B) - (A \cap C) \)
**Solution:** The first statement is false. The easiest way to see (but not prove) this is by drawing a Venn diagram. The easiest disproof is by counter example.

**Proof.** The statement is false. Consider

\[ A = \{1, 2\} \quad B = \{1, 3\} \quad C = \emptyset \]

Then we have

\[ LHS = \{1, 2\} \cup (\{1, 3\} - \emptyset) = \{1, 2\} \cup \{1, 3\} = \{1, 2, 3\} \]
\[ RHS = (\{1, 2\} \cup \{1, 3\}) - (\{1, 2\} \cup \emptyset) = \{1, 2, 3\} - \{1, 2\} = \{3\} \]

Since \( LHS \neq RHS \), the statement is false.

The second statement is true, but takes some proving. I thought up 2 styles of proof. The first one relies on knowing the set laws we saw in class.

**Proof.** Let \( A, B, C \) be sets. Then start with the RHS

\[(A \cap B) - (A \cap C) \equiv (A \cap B) \cap (A \cap \overline{C}) \quad \text{defn of set minus}\]
\[\equiv (A \cap B) \cap (A \cup C) \quad \text{De Morgan}\]
\[\equiv ((A \cap B) \cap \overline{A}) \cup ((A \cap B) \cap \overline{C}) \quad \text{Distributive law}\]
\[\equiv ((A \cap \overline{A}) \cap B) \cup (A \cap (B \cap \overline{C})) \quad \text{Associative + Commutative laws}\]
\[\equiv \overline{\emptyset} \cup (A \cap (B \cap \overline{C})) \quad \text{defn of set minus}\]
\[\equiv \emptyset \cup (A \cap (B \cap \overline{C})) \quad \text{clean things up}\]
\[\equiv A \cap (B \cap \overline{C}) \]

Hence the \( RHS \) is equivalent to the \( LHS \).

And a different style of proof, more like that we used to prove those set laws in the first place.

**Proof.** We start by showing \( LHS \subseteq RHS \) and then prove the converse.

- Let \( x \in LHS \). Hence we know that \( x \in A \) and \( x \in B - C \). Thus \( x \in B \) and \( x \notin C \). Since \( x \in A \) and \( x \in B \) we have \( x \in A \cap B \). And since \( x \notin C \), we have \( x \notin C \cap A \). Thus \( x \in RHS \).

- Let \( x \in RHS \). By the definition of relative-complement, \( x \in (A \cap B) \cap (A \cap \overline{C}) \). Hence \( x \in A \), \( x \in B \) and \( x \notin A \cap C \). Since \( x \notin A \cap C \), it follows that \( x \notin A \) OR \( x \notin C \).

Because we already know that \( x \in A \), it must be the case that \( x \notin C \). Since \( x \in B \) and \( x \notin C \), we have \( x \in B - C \). And because \( x \in A \), we have \( x \in LHS \) as required.

And so \( LHS = RHS \).
11. Find all natural numbers \( n \) such that
   a) \( 2^n > 2n + 1 \)
   b) \( 2^n > n^2 \)

Justify your answers using induction.

**Solution:**

a) For \( n = 1, 2 < 3 \), and for \( n = 2, 4 < 5 \). We will show that for all natural numbers \( n \geq 3, 2^n > 2n + 1 \).

Proof (by induction):

For any \( n \in \mathbb{N} \), and \( n \geq 3 \), denote by \( P(n) \) the statement that \( 2^n > 2n + 1 \).

For the base case when \( n = 3 \), \( P(3) \) holds because \( 2^3 = 8 > 7 = 2(3) + 1 \). Suppose that \( P(k) \) is true for some \( k \geq 3 \), that is, \( 2^k > 2k + 1 \). We will prove that \( P(k + 1) \) is also true, that is to show that \( 2^{k+1} > 2k + 3 \). We have that

\[
2^{k+1} = 2(2^k) > 2(2k + 1) \quad \text{(by the inductive hypothesis } 2^k > 2k + 1 \text{)}
\]

\[
2(2k + 1) = 4k + 2 = 2k + (2k + 2) \geq 2k + 3 \quad \text{(since } k \geq 3, 2k + 2 > 3 \text{)}
\]

Thus, \( 2^{k+1} > 2k + 3 \) and \( P(k + 1) \) is true. So, it follows from induction that \( P(n) \) holds for every natural number \( n \geq 3 \). Since we also showed that for \( n = 1 \) and \( n = 2 \), the inequality is not satisfied, \( \{n \in \mathbb{N} \mid n \geq 3\} \) is the set of all natural numbers such that \( 2^n > 2n + 1 \).

b) Verify the inequality for some low values of \( n \):

\[
\begin{align*}
n &= 1, 2 > 1 & n &= 2, 2^2 = 4 = 2^2 \\
n &= 3, 2^3 = 8 < 9 = 3^2 & n &= 4, 2^4 = 16 = 4^2.
\end{align*}
\]

We will show that for all natural numbers \( n \geq 5, 2^n > n^2 \).

Proof (by induction):

For any \( n \in \mathbb{N} \), and \( n \geq 5 \), denote by \( P(n) \) the statement that \( 2^n > n^2 \).

For the base case when \( n = 5 \), \( P(5) \) holds because \( 2^5 = 32 > 25 = 5^2 \). Suppose that \( P(k) \) is true for some \( k \geq 5 \), that is, \( 2^k > k^2 \). We will prove that \( P(k + 1) \) is also true, that is to show that \( 2^{k+1} > (k + 1)^2 \). We have that

\[
2^{k+1} = 2^k + 2^k > k^2 + 2k \quad \text{(by the inductive hypothesis } 2^k > 2k + 1 \text{)}
\]

\[
> k^2 + (2k + 1) \quad \text{(using part (a) that } 2^k > 2k + 1 \text{ for } k \geq 3 \text{)}
\]

\[
= (k + 1)^2
\]

Thus, \( 2^{k+1} > (k + 1)^2 \) and \( P(k + 1) \) is true. By induction, \( P(n) \) holds for every natural number \( n \geq 5 \). Since we also showed that the inequality does hold for \( n = 1 \) and does not for \( 1 < n < 5 \), we can conclude that \( \{n \in \mathbb{N} \mid n \geq 5 \text{ or } n = 1\} \) is the set of all natural numbers such that \( 2^n > n^2 \).
Remark: The value $k$ for which we assume $P(k)$ is true, is always at least the value of the base case (even though it may or may not be stated in the proof). Here, the fact that $k \geq 5$ in part (b) is important because it allows us to use the statement in part (a), which only holds if $k \geq 3$.

12. Let $n$ and $m$ be two integers. Prove that if $n \equiv 1 \pmod{2}$ and $m \equiv 3 \pmod{8}$ then $n^2 - 3m \equiv 0 \pmod{8}$.

Solution:
Let $n$ and $m$ be two integers. If $n \equiv 1 \pmod{2}$ and $m \equiv 3 \pmod{8}$ then $n = 2k + 1$ and $m = 8l + 3$ for some integers $k$ and $l$. Therefore we have $n^2 - 3m = 4k^2 + 4k + 1 - 24l - 9 = 4k^2 + 4k - 24l - 8 = 4k(k + 1) - 8(3l + 1)$. Since $k(k + 1)$ is the product of two consecutive integers it is divisible by two and we are done.

13. Prove that $1^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$.

Solution: $n = 1$: LHS = $1^3 = 1$, RHS = $1^2 = 1$. True. Assume $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$. For $k + 1$:

$$LHS = (1 + \cdots + k)^2 + (k + 1)^3 \quad \text{(by induction hypothesis)}$$

$$= \left(\frac{k(k + 1)}{2}\right)^2 + (k + 1)^3$$

$$= (k + 1)^2 \left(\frac{k^2}{4} + k + 1\right)$$

$$= \frac{1}{4}(k + 1)^2(k + 2)^2$$

$$= \left(\frac{(k + 1)(k + 2)}{2}\right)^2$$

$$= (1 + 2 + \cdots + k + (k + 1))^2$$

$$= RHS$$

By mathematical induction, we are done.

14. Use induction to prove that $81 \mid (10^{n+1} - 9n - 10)$ for every non-negative integer $n$.

Solution: We prove the statement by mathematical induction.

- Base case. When $n = 0$ we have $(10 - 0 - 10) = 0$ and so $81|(10^1 - 9 \cdot 0 - 10)$ as required.
Inductive step. Assume that $81|(10^{k+1} - 9k - 10)$. Hence there is some $q \in \mathbb{Z}$ so that

$$10^{k+1} - 9k - 10 = 81q$$

Now we must show the case $n = k + 1$ and to do so we consider $810q$.

$$810q = 10^{k+2} - 90k - 100$$

and now consider

$$10^{k+2} - 9(k + 1) - 10 = (810q + 90k + 100) - 9k - 9 - 10 = 810q + 81k - 81 = 81(10q + k - 1)$$

Since $10q + k - 1 \in \mathbb{Z}$, we have that $81|(10^{k+2} - 9(k + 1) - 10)$ as required.

By mathematical induction, the statement is true for all non-negative integers.

15. Prove $(5)^{1/3}$ is irrational.

**Solution:** Assume $(5)^{1/3}$ is rational. There are $m, n \in \mathbb{N}$ s.t. $m, n$ have no common factors other than 1 and $(5)^{1/3} = m/n$. Then $5 = (m/n)^3$, and so $5n^3 = m^3$. Hence $5|m^3$, and then $5|m$, and so $m = 5k$ for some $k \in \mathbb{N}$. It follows: $5n^3 = (5k)^3 = 5^3k^3$, and then $n^3 = 5^2k^3$. Thus, $5|n^3$. So $5|n$. Therefore, 5 is a common factor of $m$ and $n$, which contradicts the choice of $m, n$. We now conclude $(5)^{1/3}$ is irrational.