• 6.2, 6.8, 6.12, 6.20

• EQ1. Show that for every integer \( n \geq 2 \)

\[
\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n + 1}{2n}.
\]

• EQ2. Show \( 5 | (9^n - 4^n) \) for \( n \in \mathbb{N} \).

• EQ3. Show that for every integer \( n \geq 4 \), we have that \( n! > 2^n \).

• EQ4. Show that for every natural \( n \in \mathbb{N} \)

\[
1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.
\]

• EQ5. Show that for every natural \( n \in \mathbb{N} \)

\[
1(1!) + 2(2!) + 3(3!) + \cdots n(n!) = (n + 1)! - 1.
\]

• EQ6. Prove the following two statements using the Principle of Mathematical Induction:
  (a) For all \( n \in \mathbb{N} \), \( 2^{2n} > n(n + 1) \).
  (b) For all \( n \in \mathbb{N} \), 3 DOES NOT divide \( 7^n + 1 \).
6.2 Prove that if $A$ is any well-ordered set of real numbers and $B$ is a nonempty subset of $A$, then $B$ is also well-ordered.

**Solution:** Assume $A \subseteq \mathbb{R}$ is well-ordered and $B$ is a nonempty subset. Given a nonempty subset $C$ of $B$, $C$ is also a nonempty subset of $A$. Since $A$ is well-ordered, $C$ has a least element. So we have showed that every nonempty subset of $B$ has a least element. Hence $B$ is well-ordered.

6.8 Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integer $n$, and then verify your formula by mathematical induction.

**Solution:** We use induction to prove

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}.$$ 

When $n = 1$, obviously both hand sides of the above identity are equal to 1; so the statement holds for $n = 1$. Assume the identity holds for $k$ a positive integer:

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{n(3k - 1)}{2},$$

we use that to show the statement holds for $k + 1$:

$$1 + 4 + 7 + \cdots + (3k - 2) + [3(k + 1) - 2] = [1 + 4 + 7 + \cdots + (3k - 2)] + (3k + 1)$$

$$= \frac{n(3k - 1)}{2} + (3k + 1)$$

$$= \frac{3k^2 - k + 2(3k + 1)}{2}$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{(k + 1)(3k + 2)}{2}$$

$$= \frac{(k + 1)[(3(k + 1) - 1]}{2}.$$ 

This shows that the identity holds for $k + 1$. By the principle of mathematical induction, this proves the statement for every positive integer $n$.

6.12 Consider the open sentence $P(n) : 9 + 13 + \cdots + (4n + 5) = \frac{4n^2 + 14n + 1}{2}$, where $n \in \mathbb{N}$.

(a) Verify the implication $P(k) \Rightarrow P(k + 1)$ for an arbitrary positive integer $k$.

(b) Is $\forall n \in \mathbb{N}, P(n)$ true?

**Solution:**
(a) Assume $P(k)$ holds, namely:

$$9 + 13 + \cdots + (4k + 5) = \frac{4k^2 + 14k + 1}{2},$$

we want to prove $P(k+1)$:

$$9 + 13 + \cdots + (4k + 5) + [4(k+1) + 5] = \frac{4(k+1)^2 + 14(k+1) + 1}{2}.$$

We start from the left hand side and use $P(k)$:

$$9 + 13 + \cdots + (4k + 5) + [4(k+1) + 5] = \frac{4k^2 + 14k + 1}{2} + (4k + 9)$$

$$= \frac{4k^2 + 14k + 1 + 8k + 18}{2}$$

$$= \frac{4k^2 + 22k + 19}{2}$$

On the other hand for the right hand side of $P(k+1)$ we have:

$$\frac{4(k+1)^2 + 14(k+1) + 1}{2} = \frac{4(k+1)^2 + 2k + 1}{2} = \frac{4k^2 + 22k + 19}{2}.$$

This shows the two sides of the identity in $P(k+1)$ are equal and therefore we have proved $P(k+1)$.

(b) No! Let $n = 1$, the left hand side of the equality in $P(1)$ has only one term and is equal to 9. But for the right hand side, we have

$$\frac{4 \times 1 + 14 \times 1 + 1}{2} = \frac{19}{2}$$

which is not equal to 9.

6.20 (a)

**Solution:**

*Proof.* We use induction to prove that every set with $n$ real numbers, where $n \in \mathbb{N}$, has a largest element. Certainly the only element of a set with one element is the largest element of this set. Thus the statement is true for $n = 1$. Assume that every set with $k$ real numbers, where $k \in \mathbb{N}$, has a largest element. We show that every set with $k + 1$ real numbers has a largest element. Let $S = \{a_1, a_2, \cdots, a_{k+1}\}$ be a set with $k + 1$ real numbers. Then the subset $T = \{a_1, a_2, \cdots, a_k\}$ of $S$ has $k$ real numbers. By the induction hypothesis, $T$ has a largest element, say $a$. If $a \geq a_{k+1}$, then $a$ is the largest element of $S$. If $a < a_{k+1}$, then $a_{k+1}$ is the largest element of $S$. Thus in either case, every set with $k + 1$ real numbers has a largest element.
$S$; otherwise $a_{k+1}$ is the largest element of $S$. In either case $S$ has the largest element. \hfill \Box

EQ1

Solution:

Proof. We prove the result by induction.

* When $n = 2$ the statement is

$$1 - \frac{1}{4} = \frac{3}{4}$$

which is true.

* Assume the statement is true for $n = k$. Then

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Now multiply both sides by next term

$$\left(1 - \frac{1}{2^2}\right)\cdots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k} \cdot \left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k} \cdot \frac{k^2 + 2k + 1 - 1}{(k+1)^2}$$

$$= \frac{(k+1)(k+1)^2}{2k(k+1)^2}$$

Hence the statement is true for $n = k + 1$.

By induction the statement is true for all integer $n \geq 2$. \hfill \Box

EQ2

Solution:

Proof. We prove the result by induction

* When $n = 1$ the statement becomes “9-4 is a multiple of 5” which is true.
* Assume the statement is true for \( k = n \). Hence \( 9^k - 4^k = 5\ell \). Then

\[
5\ell = 9^k - 4^k \quad \text{multiply by 9}
\]
\[
45\ell = 9^{k+1} - 9 \cdot 4^k
= 9^{k+1} - 4 \cdot 4^k - 5 \cdot 4^k
\]

Rearrange this a little

\[
45\ell + 5 \cdot 4^k = 9^{k+1} - 4^{k+1}
\]

Since the LHS is \( 5 \cdot (9\ell + 4^k) \), the RHS is divisible by 5. Thus the statement is true for \( n = k + 1 \).

By induction the statement is true for all \( n \in \mathbb{N} \). \( \square \)

**EQ3**

**Solution:**

*Proof.* Let \( P(n) \) be the statement \( n! > 2^n \). We proceed by induction.

- When \( n = 4 \) we have \( 24 > 16 \) and so the statement is true.
- Assume \( P(k) \) is true and so \( k! > 2^k \). Then

\[
(k + 1)! = (k + 1)k! > (k + 1) \cdot 2^k
\]
\[
> 2 \cdot 2^k = 2^{k+1} \quad \text{since } k \geq 4
\]

Thus \( P(k + 1) \) is true.

By the principle of mathematical induction, the statement is true for all integers \( n \geq 4 \). \( \square \)

**EQ4**

**Solution:**

*Proof.* Let \( P(n) \) be the statement \( 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n} \). We proceed by induction.

- When \( n = 1 \) we have \( 1 \leq 2 - \frac{1}{1} \) and so the statement is true.
Assume $P(k)$ is true and so $1 + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$. Then
\[
1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}
\]
\[
= 2 - \frac{k - (k+1)^2}{k(k+1)^2} = 2 - \frac{k^2 + k + 1}{k(k+1)^2}
\]
\[
\leq 2 - \frac{k^2 + k}{k(k+1)^2} = 2 - \frac{k(k+1)}{k(k+1)^2}
\]
\[
= 2 - \frac{1}{k+1}
\]

Thus $P(k+1)$ is true.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$. □

EQ5

**Solution:**

\[1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n + 1)! - 1\]

- When $n = 1$ we have $1! = (2!) - 1 = 2 - 1 = 1$. Hence $P(1)$ is true.

- Assume $P(n)$. Hence we have

\[1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n + 1)! - 1\]

Add $(n + 1)((n + 1)!)$ to both sides:

\[1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) + (n + 1)((n + 1)!) = (n + 1)! - 1 + (n + 1)((n + 1)!)\]

\[= (n + 2)((n + 1)!) - 1\]

Thus $P(n) \Rightarrow P(n+1)$.

- Since $P(1)$ is true and $P(n) \Rightarrow P(n + 1)$, by induction $P(n)$ is true for all $n \in \mathbb{N}$.

EQ6
Solution:

(a) Let $P(n)$ be the statement: $2^{2n} > n(n + 1)$.

- First, $2^{2-1} = 4 > 1 \cdot (1 + 1) = 2$, so $P(1)$ holds.
- Suppose for contradiction that $P(k)$ is true but $P(k+1)$ is false for some $k \in \mathbb{N}$. Thus, $2^{2k} > k(k+1)$ and $2^{2(k+1)} \leq (k+1)(k+2)$. This means that

$$(k + 1)(k + 2) \geq 4 \cdot 2^{2k} > 4k(k + 1),$$

and so

$$(k + 2) > 4k.$$

This implies that $2 > 3k$ and so $k < \frac{2}{3}$, which is impossible since $k \in \mathbb{N}$. We thus have that $P(k)$ implies $P(k + 1)$ for all $k \in \mathbb{N}$.

So, by the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

(b) For all $n \in \mathbb{N}$, 3 DOES NOT divide $7^n + 1$.

Let $P(n)$ be the statement: 3 does not divide $7^n + 1$.

- First, 3 does not divide $7^1 + 1 = 8$, so $P(1)$ holds.
- Suppose for contradiction that $P(k)$ is true but $P(k+1)$ is false for some $k \in \mathbb{N}$. Hence, we have that 3 does not divide $7^k + 1$, but 3 does divide $7^{k+1} + 1$, for some $k \in \mathbb{N}$.

Then

$$7^{k+1} + 1 = 7 \cdot 7^k + 1 = 6 \cdot 7^k + (7^k + 1).$$

Since we have that 3 divides $7^{k+1} + 1$ and $6 \cdot 7^k$, then 3 must divide $7^k + 1$. We now have a contradiction, which means that 3 does not divide $7^{k+1} + 1$.

Hence, $P(k)$ implies $P(k + 1)$.

So, by the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$. 