Math 220 HW8: Induction; graphs (due on March 14/15)

1. Prove that if $u$ is a vertex of odd degree in a graph, then there exists a path from $u$ to another vertex $v$ of the graph where $v$ also has odd degree.

2. Given a connected graph $G$ and two vertices $u$ and $v$ on $G$, we define their distance $d(u, v)$ to be the length of the shortest path connecting them. Prove that $d(u, v) + d(v, w) \geq d(u, w)$ for any three vertices $u, v$ and $w$.

3. Show that any connected graph where the degree of every vertex is even has an Eulerian cycle.

4. (a) Show that every simple graph has two vertices of the same degree.
   
   (b) Show that if $n$ people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people.

5. A graph with no cycles is called a forest. Show that if a forest has $v$ vertices, $e$ edges, and $c$ connected components, then $v = e + c$.

6. A graph $G$ is called bipartite if its set of vertices can be divided into two disjoint sets $U$ and $V$ such that every edge in $G$ connects a vertex in $U$ to one in $V$. Prove that a finite graph is bipartite if and only if it contains no cycles of odd length.