Problem 1: Use the Laplace Transform table and properties of the Laplace transform to find the Laplace transforms of the following functions (You should not need to use any integration here):

(a) $f(t) = \sinh(t)$. (Comment: the “hyperbolic” trig functions are defined as follows:

$$\sinh(t) = \frac{1}{2}(e^{at} - e^{-at}) \text{ and } \cosh(t) = \frac{1}{2}(e^{at} + e^{-at}).$$

You will find that their transforms happen to look somewhat similar to the transforms of the “usual” trig functions $\sin(t)$ and $\cos(t)$.

(b) $f(t) = t^2 e^{at}$.

(c) $f(t) = e^{at}(H(t - 1) - H(t - 2))$ where $H(t)$ is the Heaviside step function. (Note: in the notation of your book, this function can be written as $f(t) = e^{at}(u_1(t) - u_2(t))$.

Problem 2: Consider the discontinuous function shown in the Figure 1.

![Figure 1](image)

(a) Express this function in terms of step function(s) multiplied by the appropriate terms.

(b) Find the Laplace Transform of this function.

(c) Consider a function of period $T = 4$ for which this is one “cycle”. Make a rough sketch of what the periodic function looks like over 2-3 cycles.

(d) What is the Laplace transform of this periodic function?
Problem 3: Find the inverse Laplace transform in each case. (Note: factoring or completing the square will be handy. Keep a copy of the Laplace transform table and the facts about those Laplace transforms nearby to help you out.)

(a) \( F(s) = \frac{1}{(s - 1)^3} + \frac{1}{s^2 + 2s - 8} \)
(b) \( F(s) = \frac{e^{-\pi s/2}}{s^2 + 9} \)
(c) \( F(s) = \frac{6}{s^3 - 9s} \)

Problem 4: Solve the following initial value problems for \( y(x) \):

(a) \( y'' + y = H(t - 3\pi), \quad y(0) = 1, y'(0) = 0 \).
(b) \( y'' - 2y' + y = e^t, \quad y(0) = 0, y'(0) = 1 \)
(c) \( y'' + 4y' + 5y = \delta(t - 2\pi), \quad y(0) = 0, y'(0) = 0 \) where \( \delta(t) \) is the Dirac delta function.

Problem 5:

(a) Use the fact that \( \mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \) to find the Laplace transforms of the functions \( t \sin(\omega t) \) and \( t \cos(\omega t) \).
(b) Use Laplace transforms to solve the initial value problem
\[ y'' + 25y = \cos(5t), \quad y(0) = 0, y'(0) = 1. \]
(Hint: in the inversion step, you will find the result of part (a) useful).

Problem 6: Consider the problem we saw before in HW2: A patient is in the hospital on intravenous medication and \( I(t) \) is the rate at which medication is infused (injected into the patient). We saw that \( c(t) \), the drug in the bloodstream satisfies the initial value problem
\[ \frac{dc}{dt} = I(t) - rc, \quad c(0) = 0. \]
where \( r > 0 \) is the rate of breakdown of the drug in the liver. Suppose the infusion rate is turned “on” at \( t = 0 \) and “off” at \( t = 1 \), so that \( I(t) = 1 \) for \( 0 \leq t \leq 1 \) and 0 otherwise.

(a) Express \( I(t) \) in terms of step function(s).
(b) Solve the above equation using the Laplace transform method with the function \( I(t) \) given in part (a).