Sept 20, 2010  Second Order linear ODE's

Mass on a spring.

\[ k \]

\[ m \]

unstretched  stretched

spring

Equilib. position.

\[ y(t) = \text{the displacement of mass from equil. position.} \]

We'll take \( y \) positive in downwards direction.

\[ \begin{align*}
\text{velo}c & \quad \text{positive} \\
\text{accel} & \quad \text{positive}
\end{align*} \]

Assume:

- mass \( m \) (kg) moves vertically
- \( y(t) \) (meters) is vertical displ. from equil.
- damping force proportional to veloc. and impedes veloc.
- spring force \( F_{\text{spring}} = -ky \) (opposite to displaum).
- forces: units of Newtons.
- mass of spring negligible.

Derive an ODE for \( y(t) \)

Use:

Newton's 2nd law \( \text{Net Force} = \text{mass} \cdot \text{accel.} \)

define veloc \( v(t) = \frac{dy}{dt} \)

accel \( a(t) = \frac{d^2y}{dt^2} \)
\[ ay(t) \text{ and its derivatives:} \]
\[ ma + \delta v + ky = 0 \]

\[ m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = 0 \] describes the displ. \( y(t) \)

ODE: 2nd order. Linear “constant coefficients” \( m, c, k \) constants.

Remark:
one solution is just \( y(t) = 0 \)
\[ y'(t) = 0 \]
\[ y''(t) = 0 \]

want to understand all possible solutions \( y(t) \) to the above ODE.
Another example of 2nd order linear ODE from Sept 8:

\[ L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \]

no applied voltage.

\[ L, R, C > 0 \] constants

\( q(t) \) - charge on capacitor

Remark: simple solution: \( q(t) = 0, i(t) = 0 \)

we want to find all other solns.

Let us discuss more generally:

\[ a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0 \]

2nd order linear ODE

\( a, b, c \) constants

\( a y'' + by' + cy = 0 \) no explicit \( f(t) \) in equation.

Solving? Try solutions of form \( y = e^{rt} \):

\[ y(t) = e^{rt}, \quad y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt} \]

plug into \( \ast \)

\[ ar^2 e^{rt} + br e^{rt} + ce^{rt} = 0 \]

recalling \( e^{rt} \neq 0 \) so can cancel \( \ast \) if

characteristic eqn.

\[ ar^2 + br + c = 0 \quad \text{quadr. eqn in } r \]

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{two possible values} \]
i.e. both \( f_1(t) = e^{rt} \) and \( f_2(t) = e^{rt} \) would satisfy ODE.

**Linear ODE ⇒ Superposition Principle**

for a linear ODE like \( \) if \( f_1(t) \) and \( f_2(t) \) are both solns, then also

\[
y(t) = C_1 f_1(t) + C_2 f_2(t)
\]

is also a soln.

**Proof:** see book.

HW 2

(\( \text{fine print} \) ⇒) "Linear superposition of \( f_1(t) \), \( f_2(t) \)"

\( \left\langle \text{"A fundamental set of solns"} \right\rangle \)

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**Example:** Solv \( 2y'' + y' - y = 0 \) (Find \( y(t) \))

**Assume** \( y(t) = e^{rt} \) plug in

\[
2r^2 e^{rt} + re^{rt} - e^{rt} = 0
\]

char. eqn

\[
2r^2 + r - 1 = 0
\]

\[
r = -1 \pm \sqrt{1 + 4.2} = -1, \frac{1}{2}
\]

**Solns:** \( e^{-t}, e^{\frac{1}{2}t} \)

All solns can be expressed as superposition.

\[
y(t) = C_1 e^{-t} + C_2 e^{\frac{1}{2}t}
\]

"general soln"

**Need two I.C.'s to find** \( C_1, C_2 \)

\( \text{e.g. } y(0) = 3 \quad y'(0) = 1 \)

Find \( C_1, C_2 \).
Using Initial Conditions to solve 2nd order, Lin. ODE

Example for Sept 20

Solve \[
2y'' + y' - y = 0 \quad \text{2nd order linear ODE}
\]

Initial conditions (I.C.'s)
\[
y(0) = 3 \quad y'(0) = 1
\]

Look for solutions of form \(y(t) = e^{rt}\). Plug \(y\) and its derivatives into the ODE, cancel common factor of \(e^{rt}\) to get

\[
2r^2 + r - 1 = 0 \quad \text{characteristic eqn.}
\]

\[
r = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{2} = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = -1, \, \frac{1}{2}
\]

Solutions:
\[
f_1(t) = e^{-t}, \quad f_2(t) = e^{\frac{1}{2}t}
\]

So \[
y(t) = C_1 e^{-t} + C_2 e^{\frac{1}{2}t} \quad \text{the general solution (includes two arbitrary constants, } C_1, C_2 \text{)}
\]

Now use I.C.'s to find \(C_1, C_2\)

\[
y(0) = 3 \quad \Rightarrow \quad 3 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \quad (1)
\]

\[
y'(0) = 1 \quad \Rightarrow \quad \text{note } y'(t) = -C_1 e^{-t} + \frac{1}{2} C_2 e^{\frac{1}{2}t}
\]

\[
y'(0) = -C_1 e^0 + \frac{1}{2} C_2 e^0 = 1 \quad \Rightarrow \quad 1 = y'(0) = -C_1 + \frac{1}{2} C_2 \quad (2)
\]

We have two equations for the two constants:

\[
\begin{align*}
(1) & \quad C_1 + C_2 = 3 \\
(2) & \quad -C_1 + \frac{1}{2} C_2 = -1
\end{align*}
\]

\[
(1) + (2) : \quad \frac{3}{2} C_2 = \frac{7}{2} \quad \Rightarrow \quad C_2 = \frac{7}{3}
\]

\[
(1) : \quad C_1 = 3 - C_2 = 3 - \frac{7}{3} = \frac{2}{3}
\]

So the solution we want is \[
y(t) = \frac{2}{3} e^{-t} + \frac{4}{3} e^{\frac{1}{2}t}
\]