Using Initial Conditions to solve 2nd order, Lin. ODE

Example for Sept 20

Solve \(2y'' + y' - y = 0\)

2nd order linear ODE

Look for solutions of form \(y(t) = e^{rt}\). Plug \(y\) and its derivatives into the ODE, cancel common factor of \(e^{rt}\) to get

\[2r^2 + r - 1 = 0\]
characteristic Eqn.

\[\begin{align*}
  r &= -1 \pm \sqrt{1 + 4 \cdot 2} \over 2.2 = -1 \pm \sqrt{9} \over 4 = -1 \pm 3 \\
    &= -1, \frac{1}{2}
\end{align*}\]

Solutions:

\[f_1(t) = e^{-t}, \quad f_2(t) = e^{t/2}\]

So

\[y(t) = C_1 e^{-t} + C_2 e^{t/2}\]

the general solution

\(\text{Include two arbitrary constants, } C_1, C_2\)

Now use IC's to find \(C_1, C_2\)

\[y(0) = 3 \quad \Rightarrow \quad 3 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \quad \text{(1)}\]

\[y'(0) = 1 \quad \Rightarrow \quad \text{note } \quad y'(t) = -C_1 e^{-t} + \frac{1}{2} C_2 e^{t/2} \quad \Rightarrow \quad\]

\[1 = y'(0) = -C_1 e^0 + \frac{1}{2} C_2 e^0 \quad \Rightarrow \quad -C_1 + \frac{1}{2} C_2 = 1 \quad \text{(2)}\]

We have two equations for the two constants:

\[
\begin{align*}
  (1) \quad & C_1 + C_2 = 3 \\
  (2) \quad & -C_1 + \frac{1}{2} C_2 = -1
\end{align*}
\]

\[(1) + (2): \quad \frac{3}{2} C_2 = 2 \quad \Rightarrow \quad C_2 = \frac{4}{3} \]

\[(1): \quad C_1 = 3 - C_2 = 3 - \frac{4}{3} = \frac{5}{3}\]

So the solution we want is

\[y(t) = \frac{5}{3} e^{-t} + \frac{4}{3} e^{t/2}\]