Systems of 1st order Linear ODEs. - Review

Recall, we are studying
\[
\begin{align*}
\frac{dx}{dt} &= a_{11}x + a_{12}y \\
\frac{dy}{dt} &= a_{21}x + a_{22}y
\end{align*}
\]
\[\Rightarrow \quad \frac{d\mathbf{x}}{dt} = \mathbf{M} \mathbf{x}, \quad \mathbf{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]

We defined $\beta = \text{Tr}(\mathbf{M}) = a_{11} + a_{22}$
\[\gamma = \det(\mathbf{M}) = a_{11} a_{22} - a_{12} a_{21}
\]

(usually)

We showed that a soln to this system is of the form
\[\frac{d\mathbf{x}}{dt} = c_1 \mathbf{v}_1 e^{r_1 t} + c_2 \mathbf{v}_2 e^{r_2 t}
\]

where $r_{1,2}$ are eigenvalues of $\mathbf{M}$ (satisfy $\det(\mathbf{M} - r \mathbf{I}) = 0$)
\[r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}
\]

and $\mathbf{v}_{1,2}$ are eigenvectors of $\mathbf{M}$ (satisfy $(\mathbf{M} - r \mathbf{I}) \cdot \mathbf{v} = 0$)
\[\mathbf{v}_i = \begin{pmatrix} r_i - a_{22} \\ a_{21} \end{pmatrix} \quad \text{(one of several possible forms for eigenvector)}
\]

We also classified the behaviour of the system according to the values of $\beta$ and $\gamma$ as follows:

- $r_{1,2}$ real, $\beta^2 - 4\gamma < 0$:
  - exponential decay
  - exponential growth

- $r_{1,2}$ real, $\beta^2 - 4\gamma > 0$:
  - oscillatory with decay ampl.
  - oscillatory with growth ampl.

- $r_{1,2}$ real, $\beta^2 - 4\gamma = 0$:
  - exponential growth and decay

- $r_{1,2}$ real, $\beta^2 - 4\gamma = 0$:
  - both exp. growth and exp. decay
In terms of the types of eigenvalues:

\[ \gamma = \sigma \pm i \mu \]

\[ \gamma_1, \gamma_2 \text{ real} \]

\[ \sigma > 0 \quad \sigma < 0 \quad \gamma_1, \gamma_2 \text{ real, } \gamma > 0 \]

\[ \beta \]

To come:

We will also see that in terms of phase-plane behaviour, same summary diagram is:
Alternate forms of eigenvectors

\[
\begin{pmatrix}
  a_{11} - r & a_{12} \\
  a_{21} & a_{22} - r
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]

\(\text{①} \quad (a_{11} - r) v_1 + a_{12} v_2 = 0\)
\(\text{②} \quad a_{21} v_1 + (a_{22} - r) v_2 = 0\)

Recall: these ops are NOT linearly indep, so can use any ONE of them

e.g. \(\text{①} \Rightarrow \text{ set } v_1 = 1 \text{ then } v_2 = \frac{r - a_{11}}{a_{12}}\)

get the form \(\begin{pmatrix} 1 \\ \frac{r - a_{11}}{a_{12}} \end{pmatrix}\) (we could also set \(v_2 = 1\) and find \(v_1\))

or \(\begin{pmatrix} a_{12} \\ r - a_{11} \end{pmatrix}\)

ALTERNATELY

\(\text{②} \Rightarrow \text{ set } v_2 = 1 \text{ then } v_1 = \frac{r - a_{22}}{a_{21}}\)

get the form \(\begin{pmatrix} \frac{r - a_{22}}{a_{21}} \\ 1 \end{pmatrix}\) or \(\begin{pmatrix} r - a_{22} \\ a_{21} \end{pmatrix}\)