Convolutions, Laplace Transforms, and the Transfer function

Let \( f(t) \) and \( g(t) \) be two functions. Then we define a special kind of product, called a convolution of \( f \) and \( g \) as follows:

\[
f \ast g = \int_0^t f(t - \tau) g(\tau) d\tau = \int_0^t f(\tau) g(t - \tau) d\tau
\]

For the geometric intuition behind this definition, see excerpt from Wikipedia (over). \( f \ast g \) is a kind of weighted average of one function with respect to (a reflected copy of) another function.

Convolutions are applied extensively in digital signal processing, and other applications. We restrict attention to their use in the context of Laplace Transforms.

Properties:

\[
f \ast g = g \ast f
\]

\[
f \ast (g + h) = f \ast g + f \ast h
\]

\[
f \ast (g \ast h) = (f \ast g) \ast h
\]
1. Express each function in terms of a dummy variable $\tau$.

2. Reflect one of the functions: $g(\tau) \rightarrow g(-\tau)$.

3. Add a time-offset, $t$, which allows $g(t - \tau)$ to slide along the $\tau$-axis.

4. Start at $\rightarrow \rightarrow$ and slide it all the way to $\rightarrow \rightarrow$. Wherever the two functions intersect, find the integral of their product.

   In other words, compute a sliding, weighted-average of function $f(\tau)$, where the weighting function is $g(-\tau)$.

   The resulting waveform (not shown here) is the convolution of functions $f$ and $g$. If $\delta(t)$ is a unit impulse, the result of this process is simply $g(0)$, which is therefore called the impulse response.
Convolution and the Laplace transform

If \( F(s) = \mathcal{L}\{f(t)\} \) and \( G(s) = \mathcal{L}\{g(t)\} \)

then \( \mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-t)g(t)\,dt \)

\[ = \int_0^t g(t) f(t-t)\,dt \]

"Convolution of the functions \( f \) and \( g \)"

**Proof:** \( F(s) = \int_0^\infty e^{-st}f(x)\,dx \), \( \mathcal{L}\{g(t)\} = \int_0^\infty e^{-sx}g(x)\,dx \) by defn.

So their product is:

\[ F(s)G(s) = \int_0^\infty e^{-st}f(x)\,dx \int_0^\infty e^{-sx}g(x)\,dx \]

\[ = \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)g(y)\,dx\,dy \]

\[ = \int_0^\infty g(y) \left[ \int_0^\infty e^{-s(x+y)} f(x)\,dx \right] \,dy \]

let \( \xi = t - \tau \) for fixed \( \tau \) \Rightarrow \,d\xi = dt \quad \xi = 0 \text{ when } t = \tau \quad \xi = \infty \text{ when } t = \infty

\[ F(s)G(s) = \int_0^\infty g(\tau) \left[ \int_{t-\tau}^{t} e^{-st} f(t-\tau)\,dt \right] \,d\tau \]

\[ = \int_0^\infty e^{-st} \left[ \int_{t-\tau}^{t} g(\tau) f(t-\tau)\,d\tau \right] \,dt \]

\[ = \mathcal{L}\{g*f\} \]
Computing Convolutions

Examples

1. \( f(t) = \sin t \quad \quad f(\tau-t) = \sin(t-\tau) \)

\[ g(t) = \delta(t-1) = \text{unit impulse at time } 1 \]

\[ g(\tau) = \delta(\tau-1) \]

\[ f \ast g(t) = \int_0^t \sin(t-\tau) \delta(\tau-1) \, d\tau = \sin(t-1) \]

(by definition of \( \delta \)-function)

(i.e., produces value of \( \sin(t-\tau) \)
for the single value \( \tau = 1 \),
where \( \delta \) fun is "centered")

2. \( f(t) = \sin t \quad \quad g(t) = e^t \)

\[ \sin(t-\tau) = \sin(t)\cos(\tau) - \sin(\tau)\cos(t) \]

\[ f \ast g(t) = \int_0^t \sin(t-\tau)e^\tau \, d\tau = \frac{1}{2} e^t \left( \cos(t-\tau) + \sin(t-\tau) \right) \bigg|_0^t \]

\[ = \frac{1}{2} \cos(t) - \frac{1}{2} \sin(t) + \frac{1}{2} e^t \]

3. \( f(t) = \sin t \quad \quad g(t) = \cos t \)

\[ f \ast g(t) = \int_0^t \sin(t-\tau) \cos(\tau) \, d\tau = \frac{1}{4} \cos(t-2\tau) + \frac{1}{2} \sin(t) \bigg|_0^t \]

\[ = \frac{1}{2} t \sin t \]

4. \( f(t) = \sin t \quad \quad g(t) = t \cos t \)

\[ f \ast g(t) = \int_0^t \sin(t-\tau) \cdot \tau \cos(\tau) \, d\tau = \frac{1}{4} \left( \sin(t) + t^2 \sin(t) : + t \cos(t) \right) \]

Then a miracle happened (Maple)

2nd proof pow!

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For example, some lengthy and arduous trig identities and IBP.
Motivation for the usefulness of Convolution Theorem.

Suppose we were asked to solve the I.V.P.
\[
y'' + y = t \cos t \quad y(0) = 0 \quad y'(0) = 0
\]

From previous experience, we expect to see terms such as \( t(At+B)(\cos t) \) and \( t(At+B)(\sin t) \) in such a case.

\[
(F(s) - sF'(0) - f(0) + F(0)) = \mathcal{L}\{t \cos t\}
\]

\[
(s^2+1)F(s) = \mathcal{L}\{t \cos t\}
\]

\[
\mathcal{L}\{t \cos t\} = \frac{d}{ds} \left( \frac{9}{s^2+1} \right) = \frac{9}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}
\]

We obtain the following algebraic formula for \( F(s) \):

\[
F(s) = \frac{s^2-1}{(s^2+1)^3} = \frac{2s}{25} \frac{(s^2+2)}{(s^2+1)^3} = \frac{1}{25} \left( \frac{2s(s^2+2)}{(s^2+1)^3} \right)
\]

The problem (always) is how to invert it to find \( y(t) \). Looking at the table:

\[
\begin{align*}
\mathcal{L}\{t \cos t\} & = \frac{d}{ds} \left( \frac{9}{s^2+1} \right) = \frac{9}{s^2+1} - \frac{s^2-1}{(s^2+1)^2} \\
\mathcal{L}\{t \sin t\} & = \frac{d}{ds} \left( \frac{1}{s^2+1} \right) = \frac{1}{s^2+1} - \frac{1}{(s^2+1)^2} \\
\mathcal{L}\{t^2 \cos t\} & = \frac{d^2}{ds^2} \left( \frac{s}{s^2+1} \right) = \frac{2s(s^2-3)}{(s^2+1)^3} \\
\mathcal{L}\{t^2 \sin t\} & = \frac{d^2}{ds^2} \left( \frac{3s^2-1}{s^2+1} \right) = \frac{2(3s^2-1)}{(s^2+1)^3}
\end{align*}
\]

\[
F(s) = \frac{3s^2 - s^2 + 1}{(s^2+1)^3} = \frac{1}{3} \frac{(s^2-1-2)}{(s^2+1)^3} = \frac{2}{3} \frac{(3s^2-1)}{(s^2+1)^3} - \frac{1}{3} \frac{1}{(s^2+1)^3}
\]

\[
= \frac{As+B}{(s^2+1)^3} + \frac{Cs+D}{(s^2+1)^2}
\]

ugly ... look for another way.

Motivation for Convolution. Thus...
Let us look at such a problem again, more generally and with Convolutions in mind.

Solve the IVP \( y'' + y = g(t) \) \( y(0) = 1, \ y'(0) = 1 \)

We may be interested in various kinds of forcing functions.

\[
\left[ sF(s) - sy(0) - y'(0) \right] + F(s) = G(s) = \mathcal{L}\{g(t)\}
\]

\[
(s^2 + 1) F(s) - s - 1 = G(s)
\]

\[
F(s) = \frac{s + 1}{s^2 + 1} + \frac{G(s)}{s^2 + 1}
\]

This part is due to initial conditions

\[
y(t) = \mathcal{L}^{-1}\{f(s)\} = \cos t + \sin t + \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \cdot G(s) \right\}
\]

\[
= \frac{S}{s^2 + 1} + \frac{1}{s^2 + 1}
\]

\[
\int_0^t \sin(t - \tau) g(\tau) d\tau
\]

It can be easier to calculate this integral than to try to invert this term by other methods.

Now we can "easily" apply this to various forcing functions.

Examples: \( g(t) = \cos t \)

We already computed \( \int_0^t \sin(t - \tau) \cos \tau d\tau = \frac{1}{2} t \sin t \)

so \( y(t) = \cos t + \sin t + \frac{1}{2} t \sin t \) (resonance case)

\( g(t) = t \cos t \)

left as exercise (see example 4 on the "computing convol" page)
Now let us think even more generally about our problems (and "philosophically")

\[
\begin{align*}
\text{Input} & \quad \xrightarrow{\text{System}} \quad \text{of interest} \quad \xrightarrow{\text{output}} \quad y(t) \\
(\text{forcing function} \quad g(t)) & \quad \xrightarrow{(\text{has properties determined by bunch of F constants})} \\
\end{align*}
\]

We want to know how the output \( y(t) \) is related to the input \( g(t) \).

The "system" could be an LRC circuit, a spring-mass system, a (linear) pendulum, etc. etc. We've studied many systems that satisfy

\[
a y'' + b y' + c y = g(t) \quad y(0) = y_0, \quad y'(0) = y_0'
\]

\( a, b, c \) are properties of the system (e.g. mass, spring constant, etc etc.)
\( y(0), y'(0) \) are initial configuration of system
\( g(t) \) is the input
\( y(t) \) is the desired (prediction for) the output.

By Laplace Transforms, we can rewrite this I.V.P. as

\[
(a s^2 + b s + c) F(s) - (a s + b) y_0 - a y_0' = G(s) = \mathcal{L}\{g(t)\}
\]

or

\[
F(s) = \frac{(a s + b) y_0 + a y_0'}{(a s^2 + b s + c)} + \frac{1}{a s^2 + b s + c} \cdot G(s)
\]

let us call these \( \Phi(s) \) and \( \Psi(s) \)
* $\Phi(s)$ comes from the initial card's and homog. problem, that is

$$\Phi(s) = \mathcal{L}\{\Phi(t)\} \text{ where } \Phi(t) \text{ is a soln to } \ a\ y'' + b\ y' + c\ y = 0 \quad y(0) = y_0, \ y'(0) = y_0'$$

$\Psi(s)$ comes from the forcing fun., i.e. nonhomog probl. but with trivial initial card's., that is,

* $\Psi(s) = \mathcal{L}\{\Psi(t)\} \text{ where } \Psi(t) \text{ is a soln to } \ a\ y'' + b\ y' + c\ y = g(t) \quad y(0) = 0, \ y'(0) = 0$

* $F(s) = \Phi(s) + \Psi(s)$

* Inverting will lead to

$$y(t) = \mathcal{L}^{-1}\{\Phi(s)\} + \mathcal{L}^{-1}\{\Psi(s)\} = \Phi(t) + \Psi(t)$$

Now look at $\Psi(s)$. Recall from last page that

$$\Psi(s) = \frac{1}{as^2 + bs + c} \cdot G(s)$$

Call it $H(s)$.

$$\Psi(s) = H(s) \cdot G(s)$$

This is known as the transfer function and it depends only on the system properties.
By the convolution theorem

\[ \mathcal{F}(s) = H(s) \cdot G(s) \iff \mathcal{F}(t) = \int_0^t h(t-\tau) g(\tau) \, d\tau \]

where \( h(t) = \mathcal{L}^{-1}\{H(s)\} \)

Thus, the part of the 'output' that results from the input can be considered as a convolution of \( g(t) \) with a special function \( h(t) \).

It turns out that \( h(t) \) (which, recall, is \( \mathcal{L}^{-1}\{H(s)\} \)) is the \textbf{impulse response} of the system. (i.e. the output obtained for a unit impulse input.)

\[ ay'' + by' + cy = S(t) \quad y(0) = 0 \quad y'(0) = 0 \]

\[ (\alpha s^2 + \beta s + \gamma) \cdot F(s) = e^s \iff 1 \]

\[ F(s) = \frac{1}{(\alpha s^2 + \beta s + \gamma)} \iff \text{Exactly how we defined } H(s) \]

\[ = H(s) \]

Thus \( y = h(t) = \mathcal{L}^{-1}\{H(s)\} \) is the solution to this impulse-driven system.