Closed book examination

Last Name: _______________ First: __________ Signature __________

Student Number ________________

Special Instructions:
- Be sure that this examination has 13 pages. Write your name on top of each page.
- You are allowed to bring into the exam one $8\frac{1}{2} \times 11$ formula sheet filled on both sides. No calculators or any other aids are allowed.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBC card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
  (a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  (b) speaking or communicating with other candidates; and
  (c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.
1. Solve the initial value problem:

\[(e^x \sin x)y' = -1 - (e^x \cos x)y,\]

with \(y(\pi/2) = e^{-\pi/2}\).

\[
\left( e^x \sin x \right) y' + \left( e^x \cos x \right) y = -1
\]

\[
y' + \frac{\cos x}{\sin x} y = -\frac{1}{e^x \sin x}
\]

Integrating factor

\[
\lambda(x) = \exp \left\{ \int \frac{\cos x}{\sin x} \, dx \right\} = \exp \left\{ \int \frac{1}{u} \, du \right\} = \exp \ln u
\]

\[
u = \sin x
\]

\[
\frac{d}{dx} \left[ y \sin x \right] = -e^{-x}
\]

\[
y \sin x = -\int e^{-x} \, dx + C
\]

\[
y = \frac{1}{\sin x} \left[ e^{-x} + C \right]
\]

\[
y(\pi/2) = e^{-\pi/2} = \frac{1}{\sin \pi/2} \left[ e^{-\pi/2} + C \right]
\]

\[
e^{-\pi/2} = 1 \cdot \left[ e^{-\pi/2} + C \right]
\]

\[
c = 0 \quad y = \frac{e^{-x}}{\sin x}
\]
2. Find all solutions of the differential equation

$$(x + 1)^2 y' + (x + 1)e^{-y} = 0.$$

$$\frac{dy}{e^{-y}} = -\frac{(x+1)}{(x+1)^3} \, dx$$

$$\int e^y \, dy = \int -\frac{1}{(x+1)^2} \, dx + C$$

$$e^y = \int -u^{-2} \, du + C \quad u = x + y$$

$$= -\frac{1}{u} + C \quad u = x + y$$

$$= \frac{1}{x+1} + C$$

Note!

$$y = \ln \left[ C + \frac{1}{x+1} \right]$$
[10] 3. Consider the differential equation

\[ y'' + p(t)y' + q(t)y = 0 \quad (*) \]

where \( p \) and \( q \) are continuous functions for all \( t \).

(a) Can \( y(t) = \sin(t^2) \) be a solution on an interval containing \( t = 0 \) of the differential equation \((*)\)? Explain your answer.

\[
\begin{align*}
  y(t) &= \sin(t^2) \quad \text{then} \quad y'(t) = 2t \cos(t^2) \\
  y''(t) &= 2(\cos(t^2) - 4t^2 \sin(t^2)) \\
  &= 2\cos(t^2) - 4t^2 \sin(t^2) \\
  \text{at} \ t=0 \quad y(0) = 0, \ y'(0) = 0, \ y''(0) = 2
\end{align*}
\]

but \((*)\) implies \( y'' = 0 \) so no, this is inconsistent.

(b) Calculate the Wronskian of \( t \) and \( t^2 \).

\[
W = \det \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} = 2t^2 - t^2 = t^2
\]

(c) Can \( t \) and \( t^2 \) both be solutions of the same differential equation \((*)\)? Explain clearly.

We notice that at \( t=0 \) Wronskian \( W=0 \), so this cannot be a fundamental set on any interval that includes \( t=0 \). But there is even more to say. Suppose there are both solutions of \((*)\). Then

\[
\begin{align*}
  t \text{ is a soln:} & \quad 0 + p(t) + q(t)t = 0 \quad (1) \\
  t^2 \text{ is a soln:} & \quad 2 + 2tp(t) + q(t)t^2 = 0 \quad (2)
\end{align*}
\]

(1) \( \Rightarrow p(t) = -q(t)t \) now substitute (2) to get

\[
2 + (-2t^2q(t)) + q(t)t^2 = 0 \\
2 - 2t^2q(t) = 0 \\
t^2q(t) = 1 \quad \Rightarrow \quad q(t) = \frac{1}{t^2}
\]

So on any interval including \( t=0 \), these 2 solutions lead to the coefficient \( q(t) \) being undefined at \( t=0 \). So this would be a contradiction.

(But on intervals that do not include \( t=0 \), there is no problem.)
4. Use the method of undetermined coefficients to find the general solution of

\[ y'' + y = \cos t. \]

**Solution to homogeneous equation:**

\[ y'' + y = 0 \]

**Characteristic Equation:**

\[ r^2 + 1 = 0 \]

\[ r = \pm i \]

**Solutions:**

\[ y(t) = C_1 \cos t + C_2 \sin t \]

**Guess for particular solution of nonhomogeneous problem:**

\[ Y_p(t) = t (A \cos t + B \sin t) \]

\[ Y_p'(t) = t (-A \sin t + B \cos t) + (A \cos t + B \sin t) \]

\[ Y_p''(t) = t (-A \cos t - B \sin t) + 2t (-A \sin t + B \cos t) + (-A \sin t + B \cos t) \]

\[ = -t (A \cos t + B \sin t) - 2A \sin t + 2B \cos t \]

\[ Y_p''(t) + Y_p'(t) = \cos t \]

\[ [t (A \cos t + B \sin t) - 2A \sin t + 2B \cos t] + t (A \cos t + B \sin t) = \cos t \]

\[ -2A = 0 \]

\[ A = 0 \]

\[ 2B = 1 \]

\[ B = \frac{1}{2} \]

**Particular Solution:**

\[ Y_p(t) = \frac{1}{2} t \sin t \]

**General Solution:**

\[ y(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t \]
5. Consider the initial value problem
\[ y'' - 3y' + 2y = g(t), \quad y(0) = 1, \ y'(0) = 1 \]
where
\[ g(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{if } t \geq 1.
\end{cases} \]

(a) Compute the Laplace transform \( \mathcal{L}\{g(t)\} \).

(b) Use the method of Laplace transforms to solve the initial value problem.
\[
F(s) \left( s^2 - 3s + 2 \right) - s - 1 + 3 = \mathcal{L}\{f(t)\}
\]
\[
F(s) = \frac{s-2}{s^2-3s+2} + \frac{\mathcal{L}\{g(t)\}}{(s^2-3s+2)}
\]
\[
= \frac{s-2}{(s-2)(s-1)} + \frac{1}{(s-2)(s-1)} \cdot \left( \frac{1-e^{-s}}{s} \right)
\]
\[
= \frac{1}{(s-1)} + \frac{1}{s(s-1)(s-2)} \left( 1 - e^{-s} \right)
\]

See partial fraction below.

\[
F(s) = \frac{1}{s-1} + \left( \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2} \cdot \frac{1}{s-2} \right) (1-e^{-s})
\]

\[
y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2} + \frac{1}{2} e^{2t} - \left( \frac{1}{2} - e^{t} + \frac{1}{2} e^{2t} \right) \cdot U_{1}(t)
\]
\[
= \frac{1}{2} \left( 1 + e^{2t} \right) - U_{1}(t) \left[ \frac{1}{2} - e^{t} + \frac{1}{2} e^{2(t-1)} \right]
\]

\[
\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad \text{Partial Fractions}
\]

\[
l = A(s-1)(s-2) + B(s-2) + C s(s-1) \]
\[
\begin{align*}
S \to 0: & \quad 1 = A(-1)(-2) \quad A = \frac{1}{2} \\
S \to 1: & \quad 1 = B \cdot 1 (-1) \quad B = -1 \\
S \to 2: & \quad 1 = C \cdot 2 (2-1) \quad C = \frac{1}{2}
\end{align*}
\]
\[
\text{Partial Fractions} = \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2} \cdot \frac{1}{s-2}
\]
6. Consider the following $2 \times 2$ matrix with real coefficients

$$A = \begin{pmatrix} 3 & 1 \\ 0 & a \end{pmatrix},$$

and consider the system of differential equations

$$X'(t) = AX(t),$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(a) Write the determinant $|A - \lambda I|$ in factored form and then find the eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $A$ in terms of $a$.

(b) In the case where $a \neq 3$, find the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$.

(c) If $a \neq 3$, find two solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, in terms of $a$, so that $\{X^{(1)}, X^{(2)}\}$ forms a fundamental set of solutions for (1).

(d) Deduce the general solution of (1) in the case $a \neq 3$.

(e) What is/are the eigenvalue(s) of $A$ in the case where $a = 3$? Find, in this case, fundamental solutions $X^{(1)}(t)$ and $X^{(2)}(t)$, and then the general solution of (1).

\(6a\) Eigenvalues: $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 3 - \lambda & 1 \\ 0 & a - \lambda \end{pmatrix} = 0 = (3 - \lambda)(a - \lambda)$$

Eigenvectors $r_1 = 3$, $r_2 = a$ (also called here $\lambda_1$, $\lambda_2$)

(Notes: since matrix is diagonal, we could have just written this directly)

(b) Eigenvectors:

$$\begin{pmatrix} 3 - \lambda & 1 \\ 0 & a - \lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3 - \lambda)v_1 + v_2 = 0$$

Let $v_1 = 1$ then $\int_0^t e^{(3 - \lambda)t} dt = \frac{1}{3 - \lambda} t - \frac{1}{(3 - \lambda)^2}$

$$v_2 = -3 + \lambda$$

Correspondingly $r_1 = 3$ is $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 + \lambda \end{pmatrix}$

$$r_2 = a$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -3 + a \end{pmatrix}$$

(c) $X_1(t) = \vec{v}_1 e^{3t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}$, $X_2 = \vec{v}_2 e^{at} = \begin{pmatrix} 1 \\ -3 + a \end{pmatrix} e^{at}$

will be a fundamental set of solutions as long as $a \neq 3$. 
6. (d) General solution is then \( \vec{x}(t) = C_1 \vec{x}_1(t) + C_2 \vec{x}_2(t) \) provided \( \lambda \neq \beta \)

i.e. \[ \vec{x}(t) = C_1 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{\beta t} + C_2 \left( \begin{array}{c} 0 \\ \lambda \end{array} \right) e^{\lambda t} \]

6.(c) If \( \lambda = \beta \), have only one solution, \( \vec{x}(t) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{3t} \) (case of repeated roots)

Form 2nd solution \( \vec{x}_2(t) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) t e^{3t} + \vec{\phi} e^{3t} \)

we can find only one eigenvector \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) here, so use the method shown in repeated roots lecture

then \( \vec{\phi} \) satisfies

\[ \vec{v} = (M - \lambda I) \vec{\phi} \]

where \( \vec{v} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) is the known eigenvector

\[ \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \cdot \left( \begin{array}{c} \beta \\ \rho \end{array} \right) \Rightarrow \beta = 1, \rho \) arbitrary \]

\[ \vec{x}_2 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) t e^{3t} + \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{3t} \]

so \( \vec{\phi} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \)

So fundamental set is

\[ \vec{x}_1 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{3t} \quad \vec{x}_2 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{3t} \]

general solution

\[ \vec{x}(t) = C_1 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{3t} + C_2 \left[ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) t e^{3t} + \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{3t} \right] \]

(a) Transform the above equation into a system of first order differential equations, and write it in matrix form $X'(t) = AX(t)$.

(b) Find two real-valued solutions $X^{(1)}(t)$ and $X^{(2)}(t)$ that form a fundamental set of solutions to the system $X' = AX$ from part (a). What is the general solution of the system?

(c) Describe the behaviour of the solutions as $t \to \infty$.

(a) There are several ways to proceed here. Note that the 2nd order ODE has char. eqn $r^2 - 4r + 13 = 0$ and its eigenvalues are $r = 2 \pm 3i$ (easy to show). The corresponding matrix for system would have same char. eqn so $\beta = -4 = \text{Trace } M$ and $\gamma = 13 = \det(M)$. Any such matrix would work so pick $M = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ for example. Then $\frac{dX}{dt} = MX$ is equivalent to the above 2nd order ODE. \( \text{Or: we could define } \begin{cases} \dot{v} = y' \\ \dot{u} = y \end{cases} \text{ and sub into ODE to get a system } \begin{cases} \dot{u} = v \\ \dot{v} = -13v + 4u \end{cases} \)

(b) Eigenvectors (as before): $r = 2 \pm 3i$

Eigenvectors: $\begin{pmatrix} 2-r & -3 \\ 3 & 2-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-r)v_1 - 3v_2 = 0$ let $v_1 = 1$ then $v_2 = \frac{2-r}{3}$

$\begin{pmatrix} 2-r \\ 3 \end{pmatrix}$ \( \text{so } v = \begin{pmatrix} 1 \\ \frac{2-r}{3} \end{pmatrix} \)

$v_{1,2} = \begin{pmatrix} 1 \\ \frac{2-r}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{i} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \alpha \pm \beta i$

$e^{rt} = e^{2t} (\cos(3t) \pm i \sin(3t))$

Now build up real valued solutions $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{2t} (\begin{pmatrix} a \cos 3t - b \sin 3t \\ b \cos 3t + a \sin 3t \end{pmatrix})$, $\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = e^{2t} (\begin{pmatrix} 3a \cos 3t - 3b \sin 3t \\ 3b \cos 3t + 3a \sin 3t \end{pmatrix})$

Get $\dot{X}(t) = c_1 u(t) + c_2 v(t)$, etc.

(c) as $t \to \infty$ $u(t)$ and $v(t)$ are both exponentially increasing oscillations. (i.e. cycles with amplitudes that grow).