Closed book examination

Last Name: _______________ First: ___________ Signature _______________

Student Number ___________________

Special Instructions:
- Be sure that this examination has 11 pages. Write your name on top of each page.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
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- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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[10] 1. Find all solutions of $y' - 2xy^2 = 0$. What kind of problem is this?

First or Second order?

Linear or Nonlinear?

Indep variable is $x$ (we used $t$ a lot)

One derivative only $\implies$ first order

$y^2$ term $\implies$ Nonlinear!!

Only hope is if we can separate variables

$$\frac{dy}{dx} = 2x y^2$$

$$\frac{dy}{y^2} = 2x \, dx$$

yes! it'll work

$$\int y^{-2} \, dy = \int 2x \, dx + C$$

$-y^{-1} = x^2 + C_1$

$-\frac{1}{y} = x^2 + C$

$y = -\frac{1}{x^2 + C}$

integrate carefully!

(power rule)

$\int y^n \, dy = \frac{y^{n+1}}{n+1}$

Don't forget this arbitrary integration constant at this step (Not later on!!)
2. Solve the initial value problem \( xy' = x^3 - 2y \), \( y(1) = 0 \).

- Note \( y = y(x) \) dependent variable appears only in \( y' \) or \( 2y \) terms.
- Problem is a linear ODE (despite \( x^3 \), since \( x \) is indep. var.)
- Non-constant coeffs but linear \( \Rightarrow \) try integrating factor
  - First put in standard form
    \[
    y' + \frac{2}{x} y = x^2 \\
    \left[ b(x) = \frac{2}{x} \right]
    \]
  - Integrating factor:
    \[
    \mu(x) = \exp \int b(x) \, dx = \exp \int \frac{2}{x} \, dx = \exp (2 \ln x) \\
    \rightarrow \frac{1}{x^2} \\
    \mu(x) = \exp \ln x^2 = e^{\ln x^2} = x^2
    \]
  - Simplify!!
  - Otherwise have a mess.
  - \( e \) and \( \ln \) are inverse functions.
  - So expression simplifies!

\[
\mu(x) \left[ y' + \frac{2}{x} y \right] = \mu(x) x^2 \\
\frac{d}{dx} [\mu(x) y] = \mu(x) x^2 \\
\frac{d}{dx} [x^2 y] = x^2 \cdot x^2 = x^4 \\
x^2 y = \int x^4 \, dx + C \\
\]

Use \( y(1) = 0 \) to find \( C \) at the end.
[20] 3. Consider the initial value problem

\[ y'' + ay' + by = 0, \quad y(0) = 3, \quad y'(0) = 5. \]

The differential equation has as a fundamental set of solutions \( \{y_1(t), y_2(t)\} \), where \( y_1(t) = e^{-t} \).

(a) Solve for \( y_2(t) \).

(b) Determine the values of the constants \( a \) and \( b \).

(c) Solve the initial value problem.

They tell me one soln is \( y_1(t) = e^{-t} \).

I don't know \( y_2(t) \), but I'm going to expect it to be something like \( e^{rt} \) (because of 2nd order lin. ODE above).

Wronskian... hmm, let's see now... isn't that

\[ W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \]

Then

\[ W = \begin{vmatrix} e^{-t} & e^{rt} \\ -e^{-t} & re^{rt} \end{vmatrix} = e^{rt} - (-e^{-t}ere^{rt}) = e^{rt}(r+1) = e^{t(r-1)}(r+1) \]

Oh, OK! \( 4e^{2t} = (r+1)e^{(r-1)t} \) so \( r+1 = 4 \) and \( r = 3 \).

Thus \( y_2(t) = e^{3t} \).

(b) From (a) I know that \( r = -1, r = 3 \) gotta be roots of char. eqn.

i.e. \( (r+1)(r-3) = 0 \) is char. eqn.!!

\[ r^2 - 2r - 3 = 0 \]

so \( a = -2, \ b = -3 \)

(c) \( y_1(t) = e^{-t}, \ y_2(t) = e^{3t} \) \( \Rightarrow \) gen. soln

\[ y(t) = c_1 e^{-t} + c_2 e^{3t} \]

I can use IC's to find \( c_1, c_2 \).
4. The homogeneous differential equation

\[ t^2y'' - 2ty' + 2y = 0, \]

defined over the open interval \( 0.5 < t < 2 \), has a non-trivial solution \( y_1 = t^2 \).

(a) Use reduction of order to find a second solution \( y_2 \).

(b) Show that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

(c) Find the particular solution that satisfies the initial conditions \( y(1) = 3 \) and \( y'(1) = 4 \).

We did not include this topic in our discussions this term.

(Luckily, since the instructor is reasonable, we wouldn't be expected to solve this problem.)
Extra space (if needed)
5. Solve the initial value problem

\[ y'' + 2y' + 5y = f(t), \quad y(0) = 1, \quad y'(0) = -1; \]

where

\[ f(t) = \begin{cases} 0 & \text{if } t < 1; \\ 1 & \text{if } t \geq 1. \end{cases} \]

This is a unit step function \( f = u_1(t) \) or \( H(t-1) \) (Heaviside step function).

**Laplace Transform:**

\[
\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{u_1(t)\}
\]

\[
[s^2F(s) - sy(0) - y'(0)] + 2[sF(s) - y(0)] + 5F(s) = \frac{e^{-s}}{s}
\]

\[
[s^2 + 2s + 5]F(s) - s + 1 - 2 = \frac{e^{-s}}{s}
\]

\[
F(s) = \frac{s + 1}{s^2 + 2s + 5} + \frac{e^{-s}}{s(s^2 + 2s + 5)}
\]

Inverting (the hardest part)

Will denominator factor? \( \Rightarrow \) if so, don't \( \times \) (won't work in this case) \( \Rightarrow \) if not, use "complete the square"

\[
\Rightarrow s^2 + 2s + 5 = (s + 1)^2 + 4
\]

So

\[
F(s) = \frac{s + 1}{(s + 1)^2 + 4} + \frac{e^{-s}}{s[(s + 1)^2 + 4]}
\]

Need partial fractions to break this up

\[
= \frac{s}{s^2 + 4} \bigg|_{s \rightarrow s + 1} + \left( \frac{4}{5} + \frac{Bs + C}{(s + 1)^2 + 4} \right) e^{-s}
\]

Find \( A, B, C \)

(a bit more work to finish this off)

\[
= \cos(2t) \cdot e^t + A u_1(t) + \ldots
\]
Extra space (if needed)

\[ x'_1 = x_1 - x_2 \quad \text{sys of 2 1st order linear ODES.} \]
\[ x'_2 = 5x_1 - 3x_2 \]

with \( x_1(0) = 1, \ x_2(0) = 3 \). Describe the behaviour of the solution as \( t \to \infty \).

We can write it in matrix form

\[ \frac{d\mathbf{x}}{dt} = \mathbf{M}\mathbf{x} \]
\[ \mathbf{M} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \]

**Eigenvalues**

\[ \det(\mathbf{M} - r\mathbf{I}) = 0 \]
\[ \begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} = (1-r)(-3-r) + 5 = r^2 + 2r + 2 = 0 \]
\[ r = -2 \pm \sqrt{4 - 8} = -2 \pm 2i = -1 \pm i \]

**Case of complex roots!**

\[ r = \sigma \pm \mu i \]
\[ \sigma = -1 \]
\[ \mu = 1 \]

- \( \sigma \) real \Rightarrow decay
- \( \mu \) real \Rightarrow oscillations

**Eigen vectors**

\[ (\mathbf{M} - r\mathbf{I})\mathbf{v} = 0 \]
\[ \begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
\[ (1-r) v_1 - v_2 = 0 \]

Take, e.g., \( v_1 = 1 \Rightarrow v_2 = 1-r \)

\[ v_1 = (1-r) \]
\[ v_2 = (1+1+i) \]

So
\[ r_1 = -1+i \]
\[ r_2 = -1-i \]

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} + \begin{pmatrix} i \end{pmatrix} \]
\[ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} - \begin{pmatrix} i \end{pmatrix} \]

**Solve!**

\[ \mathbf{x} = \mathbf{v}_1 e^{r_1 t} + \mathbf{v}_2 e^{r_2 t} \]
\[ = e^{\sigma t} (\mathbf{a} + \mathbf{b} i) e^{\mu t} (\cos \mu t + i \sin \mu t) \]
\[ = e^{\sigma t} (\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + e^{\sigma t} (\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) i \]

**Real valued**

\[ \mathbf{X}(t) = C_1 \mathbf{v}_1(t) + C_2 \mathbf{v}_2(t) \]

**General solution**

\[ \mathbf{X}(t) = C_1 \mathbf{u}(t) + C_2 \mathbf{v}(t) \]

**Use IC's to solve for \( C_1, C_2 \)**
Extra space (if needed)
7. Find a fundamental matrix for the system of equations.

\[
x' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} x.
\]

Eigenvectors:
\[
(M - rI) \cdot \vec{v} = 0
\]

\[
\begin{pmatrix} 1 - r & -2 \\ 2 & 5 - r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
r = 3 \implies \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad -2v_1 + 2v_2 = 0
\]

Pick, e.g., \( v_1 = 1 \) then \( v_2 = -1 \) so \( \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

One solution is \( x_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{3t} \) - only one eigenvector.

Make up 2nd soln \( x_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} a \\ b \end{pmatrix} e^{3t} \)

Some work would here to find \( \begin{pmatrix} a \\ b \end{pmatrix} \) so that \( x_2 \) is a soln.

Fundamental matrix is then
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{3t} & (t+a)e^{3t} \\ -e^{3t} & (-t+b)e^{3t} \end{pmatrix}
\]

(where \( a, b \) found by subst. \( x_2 \) into ODE system and finding set of 2 epsns for \( a \) and \( b \).)