Chapter 19

Differential Equations and dimensional analysis

19.1

The velocity of an object falling under the effect of gravity with air resistance is given by:

\[ \frac{dv}{dt} = f(v) = g - kv, \quad v(0) = v_0 \]

where \( g > 0 \) is acceleration due to gravity and \( k > 0 \) is a frictional coefficient (both constant).

(a) Use separation of variables to find the function \( v(t) \).

(b) By rescaling the variables \( v \) and \( t \), show that the differential equation can be written in the dimensionless form

\[ \frac{dy}{dt} = 1 - y, \quad y(0) = y_0 \]

19.2 The chemostat

A tank of fixed volume \( V \) (in Litres) contains a bacterial culture with bacterial population \( B(t) \) (gm cells per Litre) and nutrient concentration \( c(t) \) (g nutrient per Litre). A stock solution of nutrient (at fixed concentration \( c_s \) in g/L) is pumped into the tank at a constant rate \( F \) L/min, and the well mixed fluid containing bacteria and nutrient is pumped out of the tank at the same rate (to assure that the volume of the culture does not change). The bacteria consume nutrient at rate \( g(c) \) and grow at a proportional rate (with proportionality constant \( \alpha \)). In class, some of you derived the following equations to describe this chemostat:

\[ \frac{dc}{dt} = \frac{F}{V}(c_s - c) - g(c)B \]  \hspace{1cm} (19.1)

\[ \frac{dB}{dt} = \alpha g(c)B - \frac{F}{V}B \]  \hspace{1cm} (19.2)

We also assumed that the bacteria consume nutrients at the rate

\[ g(c) = \frac{K_{max}c}{k_n + c} \]
(a) Determine the units of each of the parameters in this problem.

(b) Rescale the variables \( c = y \bar{c}, B = x \bar{B} \) and simplify the equations to show that the model can be written in the dimensionless form

\[
\frac{dy}{dt} = \alpha_2 - y - \frac{y}{1+y}x, \quad (19.3)
\]

\[
\frac{dx}{dt} = \alpha_1 \frac{y}{1+y}x - x. \quad (19.4)
\]

What are the new parameters \( \alpha_1, \alpha_2 \) in terms of the old parameters?

(c) Find the steady states of the dimensionless model.

(d) In order for a steady state to make biological sense, the value of bacteria and nutrient concentration cannot be negative (and this goes also for the dimensionless forms of these variables.) Find the condition on the parameters \( \alpha_1, \alpha_2 \) such that at steady state, the tank will contain both bacteria and nutrient. [Note: you are not asked to determine stability, only to ensure that non-negative steady state exists.]

(e) Interpret your results in terms of the original parameters and indicate how you need to adjust the chemostat to ensure that the bacterial culture will not be “washed out”.