Chapter 15

Stability and Linearization, Systems of Differential Equations

15.1 Linear approximation and stability

Consider the differential equation
\[ \frac{dx}{dt} = f(x) \]
Suppose that the steady state of this equation is \( x = X_{ss} \).

(a) Use linear approximation to express \( f(X_{ss} + \Delta x) \) in terms of the values of \( f \) and \( f' \) at \( X_{ss} \).

(b) Consider a point that is close to the steady state, i.e. let \( x = X_{ss} + \Delta x \). Substitute this into the differential equation and simplify your result to obtain a new differential equation for the deviation \( \Delta x \) from steady state.

(c) A steady state is said to be stable if small deviations from that steady state decay. Show that \( X_{ss} \) is stable if \( f'(X_{ss}) < 0 \). We will refer to this condition as the **stability criterion**.

15.2 Stability, single ODE

Consider the differential equation
\[ \frac{dx}{dt} = f(x) \]
and suppose that this equation has a stable a steady state located at \( x = 1 \). Which of the following statements could then be true?

(a) \( f(1) = 1, f'(1) = 0 \)
(b) \( f(1) = 1, f'(1) = -1 \)
(c) \( f(1) = 0, f'(1) = 1 \)
(d) \( f(1) = 0, f'(1) = -1 \)
(e) \( f(1) = -1, f'(1) = -1 \)
15.3 Single ODE’s cont’d

For each of the following single ODE’s, find all steady states and determine stability of those steady states using the stability criterion developed above. (If the stability criterion does not apply, explain why.)

(a) \( \frac{dy}{dt} = y^2 - 3y + 2 \)

(b) \( \frac{dx}{dt} = rx(1 - x), \) for \( r > 0 \)

(c) \( \frac{dx}{dt} = 3x^2(1 - x) \)

(d) \( \frac{dy}{dt} = y \)

15.4 Linear approximation, Two variables

In the following problems, a function \( f(x, y) \) depends on two variables. Use linear approximation to express \( f(X_0 + \Delta x, Y_0 + \Delta y) \) in terms of the values of \( f \) and its partial derivatives at \((X_0, Y_0)\). Simplify as much as possible. (Your answer will be in terms of \( \Delta x, \Delta y \).)

(a) \( f(x, y) = x + y \) at \( X_0 = 0, Y_0 = 0 \)

(b) \( f(x, y) = x^2 + y^3 \) at \( X_0 = 1, Y_0 = 1 \)

(c) \( f(x, y) = x - xy \) at \( X_0 = 1, Y_0 = 1 \)

(d) \( f(x, y) = x(1 - 2x - 3y) \) at \( X_0 = 0, Y_0 = 1 \)

15.5 Stability, System of ODE’s

Consider the system of ODE’s given below

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y)
\end{align*}
\]

Suppose that \((X_{ss}, Y_{ss})\) is a steady state of this system. Show that close to this steady state, the system can be approximated by the linear system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

where the coefficients \(a, b, c, d\) are partial derivatives of \(f, g\) evaluated at \((X_{ss}, Y_{ss})\).
15.6 A Linear System

Consider the linear system of ODEs
\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) = ax + by \quad (15.5) \\
\frac{dy}{dt} &= g(x, y) = cx + dy \quad (15.6)
\end{align*}
\]

This system has a single steady state, at (0, 0). Show that the partial derivatives of \( f, g \) are precisely the linear coefficients \( a, b, c, d \) in this system. Show that the system is stable if \( \beta = a + d < 0 \) and \( \gamma = ad - bc > 0 \).

15.7 Stability, Continued

Consider the system of differential equations given below,
\[
\begin{align*}
\frac{dx}{dt} &= x(1 - 2y - x) \quad (15.7) \\
\frac{dy}{dt} &= y - x \quad (15.8)
\end{align*}
\]

(a) Find all steady states of this system.

(b) Determine the stability of each of the steady states by linearizing the system about that steady state and determining the behaviour of that linear system.