Chapter 14

Vectorfields and Systems of Differential Equations

14.1 Sketching Vectorfields 1.

Draw a sketch of each of the following vectorfields

(a) \( V(x, y) = (-x, -y) \)

(b) \( V(x, y) = (-y, x) \)

(c) \( V(x, y) = (-y, -x) \)

(d) \( V(x, y) = (x + y, x - y) \)

14.2 Sketching Vectorfields 2

Draw a sketch of each of the following vectorfields. These are a little harder. You may want to consider places where either the \( x \) or the \( y \) component(s) of these vectors are zero to help organize your work.

(a) \( V(x, y) = (1 - y - x, x - y) \)

(b) \( V(x, y) = (1 - x, 2 - 2y) \)

(c) \( V(x, y) = (3 - 3x, y - 1) \)

(d) \( V(x, y) = (x - xy, xy - y) \)

Detailed Solution:

See Fig. 14.1.
Figure 14.1: Solution to Problem 14.2.
14.3

Consider the system of differential equations

\[
\frac{dx}{dt} = -x \\
\frac{dy}{dt} = y
\]

This system of equations is not really coupled. Each equation can be solved independently.

(a) Suppose that at time \( t = 0 \) the initial values are \((x_0, y_0) = (2, 0.1)\). Solve these differential equations and sketch the solutions as two curves \( x \) versus \( t \) and \( y \) versus \( t \) on the same coordinate system (time will be on the horizontal axis).

(b) Sketch the vectorfield corresponding to this set of differential equations in the \( xy \) plane and sketch the curve \((x(t), y(t))\) corresponding to the solution that you sketched in part (a).

(c) What point corresponds to the steady state of this system?

14.4

Consider the system of differential equations

\[
\frac{dx}{dt} = -y \\
\frac{dy}{dt} = x
\]

(a) Show that you can eliminate \( y \) and arrive at the single second order differential equation

\[
\frac{d^2x}{dt^2} = -x.
\]

(A second order equation is one in which the highest derivative is the second derivative.)

(b) Show that you arrive at a similar equation for \( y \) by eliminating \( x \).

(c) Check that the functions \( x = \cos(t) \) and \( x = \sin(t) \) both satisfy the second order equation in part (a).

(d) Show that the function \( x = A \cos(t) + B \sin(t) \) also satisfies the same equation for any constants \( A, B \). (This new function is called a linear superposition of the two solutions \( x = \cos(t) \) and \( x = \sin(t) \).)

Detailed Solution:

(a) This can be seen by differentiating the first equation and plugging in \( x \) for \( dy/dt \).

(b) Same idea, but differentiate the second equation and plug \(-y\) in for \( dx/dt \).

(c) This is a simple property of the derivatives of these trigonometric functions.

(d) An elementary calculation shows that this is true.
14.5 Linear Superposition

Consider the linear differential equation of second order
\[ p \frac{d^2x}{dt^2} + q \frac{dx}{dt} + rx = 0. \]
Suppose that \( x = x_1(t) \) and \( x = x_2(t) \) are two solutions. Show that the linear superposition
\[ x = Ax_1(t) + Bx_2(t) \]
is also a solution.

14.6 Reduction to a second order ODE

Consider the system of linear differential equations
\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]
Show that this system can be reduced to the single linear second order differential equation
\[
\frac{d^2x}{dt^2} - \beta \frac{dx}{dt} + \gamma x = 0.
\]
where \( \beta = a + d, \gamma = ad - bc. \)

Detailed Solution:
\[
\frac{dx}{dt} = ax + by, \quad \Rightarrow \quad \frac{d^2x}{dt^2} = a \frac{dx}{dt} + b \frac{dy}{dt} = a \frac{dx}{dt} + b(cx + dy)
\]
Now use
\[
by = \frac{dx}{dt} - ax
\]
to eliminate the remaining \( y \) term. Rearrange the result to obtain the desired equation.

14.7 Eigenvalues

Consider the linear second order differential equation
\[
\frac{d^2x}{dt^2} - \beta \frac{dx}{dt} + \gamma x = 0.
\]
(a) Show that solutions of the form
\[ x(t) = Ae^{\lambda t} \]
will satisfy this equation for any constant \( A \), provided that
\[
\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}.
\]
There are two possible values (called eigenvalues) in this formula.
(b) Show that if $\gamma < 0$ it is always true that both values are real and have opposite signs.

(c) Show that if $\gamma > 0$, $\beta < 0$ then the real part of $\lambda$ is always negative.

(d) Show that if $\gamma > \beta^2/4$ then both values of $\lambda$ are complex numbers, and that they are in fact complex conjugates. (A pair of complex conjugates are complex numbers of the form $a + bi, a - bi$.)

### 14.8 More Linear Superposition

Consider a system of linear differential equations

\[
\frac{dx}{dt} = ax + by \\
\frac{dy}{dt} = cx + dy
\]

Suppose that the eigenvalues of this system are $\lambda_1, \lambda_2$.

(a) Explain why it is then true that the solution for $x$ would have the form

\[x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.\]

(b) Find the corresponding form for the variable $y$ using the first equation. (You will have to differentiate the function in part a.)

(c) Show that the solution can be written in vector form:

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}
\]

i.e. find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

**Detailed Solution:**

(a) This follows from previous problems and from the fact that each of $e^{\lambda_2 t}, e^{\lambda_2 t}$ is a solution to the system (since the system is equivalent to a single second order DE). Thus, by linear superposition, this linear combination of functions is also a solution.

(b) From the first equation, it follows that

\[by = \frac{dx}{dt} - ax.\]

Differentiation of the solution for $x(t)$ and rearrangement will lead to

\[y(t) = \frac{1}{b} \left( A\lambda_1 e^{\lambda_1 t} + B\lambda_2 e^{\lambda_2 t} - a(Ae^{\lambda_1 t} + Be^{\lambda_2 t}).\right)\]

\[y(t) = \frac{1}{b} \left( A(\lambda_1 - a)e^{\lambda_1 t} + B(\lambda_2 - a)e^{\lambda_2 t}\right)\]

\[y(t) = \left( A\frac{(\lambda_1 - a)}{b} e^{\lambda_1 t} + B\frac{(\lambda_2 - a)}{b} e^{\lambda_2 t}\right)\]
(c) The eigenvectors will have the form

\[ \mathbf{v}_i = \left( \frac{1}{\lambda_i - \alpha} \right) \]

### 14.9 Linear Systems

Consider the system of linear differential equations

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

with coefficients given below. In each case, determine the eigenvalues and eigenvectors of the system of equations.

(a) \( a = 1, b = -2, c = -1, d = 3 \)

(b) \( a = -1, b = 3, c = -1, d = 2 \)

(c) \( a = -2, b = 1, c = -1, d = -2 \)

(d) \( a = 4, b = 1, c = 0, d = -1 \)

### 14.10 Damped Cycles 1

Suppose that a linear system of two first order differential equations has eigenvalues

\[
\lambda_{1,2} = -k \pm \omega i,
\]

where \( k, \omega > 0 \) are constants. Use the identity

\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \]

to show that solutions will occur in the form of cycles with decaying amplitude.

**Detailed Solution:**

The solutions are of the form

\[ x(t) = Ae^{\lambda t} = Ae^{(-k \pm \omega i)t}. \]

This results in

\[ x(t) = Ae^{-kt}e^{\pm i\omega t} = Ae^{-kt}[\cos(\omega t) \pm i \sin(\omega t)]. \]

Linear combinations of these will lead to the two solutions

\[ x(t) = Ae^{-kt}\cos(\omega t), \quad Ae^{-kt}\sin(\omega t). \]

These functions are oscillatory, but their amplitudes decay exponentially.
14.11 Damped Cycles 2

Consider the system of differential equations

\[
\begin{align*}
\frac{dx}{dt} & = -x + 2y \\
\frac{dy}{dt} & = -2x - y
\end{align*}
\]

Show that the solutions to this system of equations will have damped oscillations and sketch a typical solution curve in the xy plane.

14.12 Stability

The system of equations that we have explored above,

\[
\begin{align*}
\frac{dx}{dt} & = ax + by \\
\frac{dy}{dt} & = cx + dy
\end{align*}
\]

always has a steady state at \((x, y) = (0, 0)\). Argue that if \(\beta = a + d\) is negative and \(\gamma = ad - bc\) is positive, then that steady state is stable, i.e. solutions of the system will get closer and closer to this steady state. (Note: later we will make a distinction between locally and globally stable steady states, but for such linear systems these two notions are the same.)

Detailed Solution:

This fact follows from the fact that eigenvalues of the linear system are expressed in terms of the quantities \(\beta, \gamma\) as previously noted:

\[
\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}
\]

If \(\gamma > 0\) then either we have oscillations (when \(\beta^2 - 4\gamma < 0\)) in which case the fact that \(\beta < 0\) implies damping, i.e. decrease of amplitude, so that solutions approach \((0,0)\) with damped oscillations. Or else \(\beta^2 - 4\gamma > 0\), but since \(\gamma > 0\), the square root is smaller than \(\beta\), so that both \(\pm\) roots are negative. In this second case, the solutions are decaying exponentials, again approaching \((0,0)\).

14.13 Systems of differential equations

For each of the following sets of differential equations, sketch some of the solution curves in the xy plane, showing points of special interest (steady states) and the direction and type of flow. Note the relationship of this problem to Problem 14.2.

(a) \[
\frac{dx}{dt} = 1 - y - x, \quad \frac{dy}{dt} = x - y
\]
(b) \[ \frac{dx}{dt} = 1 - x, \quad \frac{dy}{dt} = 2 - 2y \]

(c) \[ \frac{dx}{dt} = 3 - 3x, \quad \frac{dy}{dt} = y - 1 \]

(d) \[ \frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = xy - y \]

14.14 A chemical reaction

A chemical X is produced at a constant rate \( R \). It is interconverted into form Y at rate \( k_1 \) and the reverse reaction occurs at rate \( k_2 \). The form Y decays at rate \( \delta \) into other products. (The constants \( R, k_1, k_2, \delta \) are all positive.) The system of equations that describes the changes in the concentrations of these chemicals, \( x(t), y(t) \) is:

\[
\begin{align*}
\frac{dx}{dt} &= R - k_1 x + k_2 y \\
\frac{dy}{dt} &= k_1 x - (k_2 + \delta) y
\end{align*}
\]

(a) Find the steady state concentrations of X and Y predicted by this model. We will refer to this steady state with the notation \( x_s, y_s \)

(b) Let \( x'(t), y'(t) \) denote deviations from steady state (not derivatives!) Consider any state close to steady state, \( x(t) = x_s + x'(t), y(t) = y_s + y'(t) \). Find the equations satisfied by the deviations from steady state. (Plug in the forms \( x(t) = x_s + x'(t), y(t) = y_s + y'(t) \) into the equations and simplify.)

(c) From the equations for \( x', y' \), in part (b) explain how we can conclude that points close to steady state will be attracted to the steady state, i.e. the deviations will decrease with time.

Detailed Solution:

(a) Steady states satisfy

\[
\begin{align*}
0 &= R - k_1 x_s + k_2 y_s \\
0 &= k_1 x_s - (k_2 + \delta) y_s
\end{align*}
\]

This is a system of linear algebraic equations that can be solved easily. The solution is \( y_s = R/\delta, x_s = (k_2 + \delta)R/\delta k_1 \)

(b) Plugging \( x(t) = x_s + x'(t), y(t) = y_s + y'(t) \) into the equations and using the fact that

\[
\begin{align*}
0 &= \frac{dx_s}{dt} = R - k_1 x_s + k_2 y_s \\
0 &= \frac{dy_s}{dt} = k_1 x_s - (k_2 + \delta) y_s
\end{align*}
\]
leads, after some simplification, to the equations for \(x', y'\)

\[
\frac{dx'}{dt} = -k_1 x' + k_2 y' \\
\frac{dy'}{dt} = k_1 x' - (k_2 + \delta)y'
\]

(c) The equations in (c) are of the form of the simple linear systems we have been exploring, with the coefficients \(a = -k_1, b = k_2, c = k_1, d = -(k_2 + \delta)\). The quantities of interest in the eigenvalue expression are then \(\beta = a + d = -(k_1 + k_2 + \delta) < 0\) and \(\gamma = ad - bc = k_1(k_2 + \delta) - k_1k_2 = k_1\delta > 0\). Since \(\beta < 0, \gamma > 0\) we know that the solutions will be decreasing. Thus the deviations from steady state will decrease with time, i.e. states close to steady state will be attracted to that steady state.

### 14.15 Predator-prey interactions

Consider the population of prey, \(x(t)\) and predators \(y(t)\) that satisfy the system of equations

\[
\frac{dx}{dt} = f(x, y) = x - xy \\
\frac{dy}{dt} = g(x, y) = xy - y
\]

Note the relationship of this problem to Problem 14.13(d).

(a) Find all steady states of the system.

(b) Linearize the system about the equilibrium (0,0) and show that this equilibrium is a saddle point, i.e. has one positive and one negative real eigenvalues.

(c) Linearize the equations about the equilibrium in the first quadrant and show that the eigenvalues are pure imaginary (i.e. complex with zero real parts).

### 14.16 Species Competition

The following set of differential equations have been used to describe the populations of two interacting species of fish:

\[
\frac{dN_1}{dt} = r_1 N_1(1 - a_1 N_1 - a_2 N_2), \quad (14.1) \\
\frac{dN_2}{dt} = r_2 N_2(1 - b_2 N_2 - b_1 N_1). \quad (14.2)
\]

where \(N_1(t), N_2(t)\) are the number of individuals of species 1 and species 2 at time \(t\), in a given lake. Assume \(r_1, r_2, a_1, a_2, b_1, b_2 > 0\) are constants.

(a) Explain what these equations are saying about the fish and their interactions.
(b) Suppose species 2 has been eliminated due to overfishing. What would happen to species 1?

(c) Find all steady states of this system of differential equations in the specific case that \( a_1 = a_2 = 1 \) and \( b_1 = b_2 = 2 \).

(d) Sketch the vector field and a few solutions in the \( N_1N_2 \) plane.

(e) Which of the steady states that you have found in part (c) is stable?

(f) What is this saying about the outcome of the competition in this case?

**Detailed Solution:**

(a) The equations can also be written
\[
\frac{dN_1}{dt} = r_1 N_1 (1 - a_1 N_1) - \beta N_1 N_2, \quad (14.3)
\]
\[
\frac{dN_2}{dt} = r_2 N_2 (1 - b_2 N_2) - \alpha N_1 N_2. \quad (14.4)
\]

In this form we can see that each population will grow logistically, while the contact between them will lead to a decline in each population (see the negative terms at the end). Thus the populations are competing, or killing one another.

(b) If species 2 is eliminated due to overfishing then \( N_2 = 0 \). In this case,
\[
\frac{dN_1}{dt} = r_1 N_1 (1 - a_1 N_1 - a_2 N_2) = r_1 N_1 (1 - a_1 N_1).
\]

This is just a type of logistic equation. The steady states are \( N_1 = 0 \) and \( N_1 = 1/a_1 \). Based on what is known about the behaviour of the logistic equation, we expect that \( N_1(t) \) will approach the "carrying capacity" of \( 1/a_1 \).

(c) In the specific case that \( a_1 = a_2 = 1 \) and \( b_1 = b_2 = 2 \), the steady states satisfy \( dN_1/dt = 0, dN_2/dt = 0 \), so
\[
\frac{dN_1}{dt} = r_1 N_1 (1 - N_1 - N_2) = 0, \quad (14.5)
\]
\[
\frac{dN_2}{dt} = r_2 N_2 (1 - 2N_2 - 2N_1) = 0. \quad (14.6)
\]

Solutions of Eqn 14.5 are \( N_1 = 0 \) or \( (1 - N_1 - N_2) = 0 \), and for Eqn 14.6, either \( N_2 = 0 \) or \( (1 - 2N_2 - 2N_1) = 0 \).

If \( N_1 = 0 \) then either \( N_2 = 0 \) or \( N_2 = 1/2 \). This gives the steady states \((N_1, N_2) = (0, 0)\) and \((N_1, N_2) = (0, 1/2)\).

If \( N_2 = 0 \), then either \( N_1 = 0 \) (considered above) or else \( N_1 = 1 \), giving the additional steady state \((N_1, N_2) = (1, 0)\).

There are no other solutions, because the two equations \((1-N_1-N_2) = 0\) and \((1-2N_2-2N_1) = 0\) represent two lines of equal slope (never intersect), so there is no other value of \((N_1, N_2)\) that satisfies these equations.
(d) See Fig. 14.2.

(e) The stable steady state is \((N_1, N_2) = (1, 0)\), as seen from Fig 14.2. This can also be deduced by carrying our a full linearization of the system about each of the steady states.

(f) In this case, if we start with some level of each of the species, eventually species 1 will win the competition, and only it will survive.