Reaction-Diffusion Systems and Turing patterns
What is a reaction-diffusion system?

Reaction
\[
\frac{\partial A}{\partial t} = f(A,B) \\
\frac{\partial B}{\partial t} = g(A,B)
\]

Example:
\[S_1 \leftrightarrow A\]
\[S_2 \xrightarrow{k_4} B\]
Shankenbery
\[B + 2A \xrightarrow{k_3} 3A\]

\[f(A,B) = k_1 - k_2 A + k_3 A^2 B\]
\[g(A,B) = k_4 - k_3 A^2 B\]

Diffusion
\[
\frac{\partial A}{\partial t} = D \frac{\partial^2 A}{\partial x^2}
\]

Units: \([D] = L^2/t\)
Alan Turing
1912-1954

- Instrumental in early computer science ("Turing Machine")
- King’s College Cambridge, 1931-34; PhD Princeton, 1938
- WWII: Bletchley Park, cryptography, deciphering German”enigma” code

- 1952: work on reaction-diffusion equations, pattern formation, and morphogenesis

Turing (1952): “The chemical basis of morphogenesis”

With “appropriate” reaction kinetics, an RD system can form spatial patterns due to the effect of diffusion.

Ingredients required for “Turing” pattern formation

\[ \frac{\partial A}{\partial t} = f(A,B) + D_A \frac{\partial^2 A}{\partial x^2} \]
\[ \frac{\partial B}{\partial t} = g(A,B) + D_B \frac{\partial^2 B}{\partial x^2} \]

- There is a stable spatially homogeneous steady state, \( f(A,B)=0, g(A,B)=0 \).
- Range of activator \(< < \) range of inhibitor

Close to that state: (Linearize eqns using Taylor Series for \( f, g \))

\[ \frac{\partial a}{\partial t} = c_{11}a + c_{12}b + D_A \frac{\partial^2 a}{\partial x^2} \]
\[ \frac{\partial b}{\partial t} = c_{21}a + c_{22}b + D_B \frac{\partial^2 b}{\partial x^2} \]

\[ \frac{D_a}{c_{11}} \ll \frac{D_b}{c_{22}} \]

“on center/off surround”
lateral inhibition: local excitation, long-range inhibition
Example: Shnakenberg RD system

Starting close to the HSS, the system evolves a spatial pattern that persists with time.

\[
\frac{\partial A}{\partial t} = f(A, B) + D_A \frac{\partial^2 A}{\partial x^2}
\]

\[
\frac{\partial B}{\partial t} = g(A, B) + D_B \frac{\partial^2 B}{\partial x^2}
\]

\[
f(A, B) = k_1 - k_2 A + k_3 A^2 B
\]

\[
g(A, B) = k_4 - k_3 A^2 B
\]
Contributors to theories of pattern formation and chemotaxis:

Hans Meinhardt    Lee Segel
Pattern formation in biology

Hans Meinhardt

\[
\frac{\partial a}{\partial t} = \frac{\rho \ a^2}{h} - \mu_a \ a + D_a \frac{\partial^2 a}{\partial x^2} + \rho_o \\
\frac{\partial h}{\partial t} = \frac{\rho \ a^2 - \mu_h \ h + D_h \frac{\partial^2 h}{\partial x^2}}{I}
\]

http://www.eb.tuebingen.mpg.de/departments/former-departments/h-meinhardt/home.html
Reaction-Diffusion systems and animal coat patterns