1. (Review) Two spherical cells are connected by a thin “neck” so that when one expands, the other contracts. Let $V_1$ and $V_2$ denote the volumes of the two cells and assume that the total volume is fixed. If the radius of cell 1 decreases at a constant rate, at what rate does the radius of the second cell change at the instant when the ratio of the radii ($r_1/r_2$) is 2?

**Solution** Let $r_1, r_2$ be the two radii. Since the cells are spherical, $V_i = (4/3)\pi r_i^3$. So

$$V_1 + V_2 = V = \text{constant} \Rightarrow \frac{dV_1}{dt} + \frac{dV_2}{dt} = 0 \Rightarrow \frac{dV_1}{dt} = -\frac{dV_2}{dt}$$

Now converting to radii

$$\frac{d(4/3)\pi r_1^3}{dt} = -\frac{d(4/3)\pi r_2^3}{dt} \Rightarrow \frac{4}{3}\pi \cdot 3r_1 \frac{dr_1}{dt} = -\frac{4}{3}\pi \cdot 3r_2 \frac{dr_2}{dt} \Rightarrow r_1 \frac{dr_1}{dt} = -r_2 \frac{dr_2}{dt}$$

We know that the radius of cell 1 decreases at a constant rate, so that $dr_1/dt = -k$. From this we can conclude that

$$\frac{dr_2}{dt} = -\frac{r_1^2}{r_2^2} k.$$

At the instant that the ratio $r_1/r_2 = 2$ we get

$$\frac{dr_2}{dt} = 4k.$$

2. A pond has two populations of amoebae, $N_1, N_2$. The populations start out with $N_1 = 100$ and $N_2 = 40$ at time $t = 0$. (a) Assuming unlimited growth, write down the differential equations satisfied by $N_1, N_2$. (b) You observe that after 1 day, the two are equal. If the growth rate of $N_2$ is $k_2 = 2$ per day, find $k_1$. (c) When will $N_2$ be 100 times larger than $N_1$?

**Solution**

$$\frac{dN_1}{dt} = k_1 N_1, \quad \frac{dN_2}{dt} = k_2 N_2, \quad N_1(0) = 100, N_2(0) = 40.$$
So, the populations obey
\[ N_1(t) = 100e^{k_1t}, \quad N_2(t) = 40e^{k_2t}. \]
At \( t = 1 \) day these are equal, so
\[ 100e^{k_1} = 40e^{k_2} \quad \Rightarrow \quad e^{k_1} = 0.4e^{k_2} = 0.4e^{2-1} = 0.4. \]
Take logarithms of both sides
\[ \ln(e^{k_1}) = \ln(0.4e^{2-1}) \quad \Rightarrow \quad k_1 = \ln(0.4) + 2\ln(e) = \ln(0.4) + 2 = 1.084. \]
Hence the growth rate of the first species is \( k_1 = 1.084 \) per day.
(c) \( N_2 \) will be 100 times \( N_1 \) when
\[ N_2 = 100N_1 \quad \Rightarrow \quad 40e^{k_2t} = 100(100e^{k_1t}) \]
We know \( k_1, k_2 \) and must solve for \( t \):
\[ e^{k_2t} = 250(e^{k_1t}) \quad \Rightarrow \quad e^{k_2t-k_1t} = 250 \quad \Rightarrow \quad (k_2 - k_1)t \]
Hence, after substituting in the values of the rate constants, we get
\[ t = \frac{\ln(250)}{(k_2 - k_1)} = \frac{\ln(250)}{(2 - 1.084)} = 6.027. \]
It will take 6 days until the 2nd species is 100 times larger.

3. Find critical points and inflection points for \( y = f(x) = x^3e^{-kx} \).

4. (Review) Consider the E coli cell described in the midterm question Q10. Suppose that the radius and height of the cell are increasing at a constant rate \( k \). Find the rate of change of the volume \( V \) and the surface area \( S \) of the cell.

\textbf{Solution} The volume of the cell is the sum of the cylindrical and spherical parts. The radius of both the cylinder and the sphere is \( r \) and the height of the cylinder is \( h \), so
\[ V = \pi r^2h + \frac{4}{3}\pi r^2. \]
The rate of increase of the volume is then
\[ \frac{dV}{dt} = \frac{d}{dt} \left( \pi r(t)^2h(t) + \frac{4}{3}\pi r(t)^3 \right). \]
Using the product and chain rules we get

\[
\frac{dV}{dt} = \left( \pi [2r(t)r'(t)h(t) + r(t)^2h'(t)] + \frac{4}{3} \cdot 3\pi r(t)^2r'(t) \right).
\]

Both the radius and the height increase at a constant rate, \( k \) so that

\[ r'(t) = k, \quad h'(t) = k. \]

Hence

\[
\frac{dV}{dt} = \left( \pi [2rk + r^2k] + 4\pi r^2k \right).
\]

This can get simplified to

\[
\frac{dV}{dt} = \pi \left( 2rk + 5r^2k \right) = \pi rk \left( 2h + 5r \right).
\]