Review for Differential Calculus: Mathematics 102

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Exercises

0.1. **Multiple Choice:**

1 : The equation of the tangent line to the function $y = f(x)$ at the point x_0 is
(a) $y = f'(x_0) + f(x_0)(x - x_0)$
(b) $y = x_0 + f(x_0)/f'(x_0)$
(c) $y = f(x) - f'(x)(x - x_0)$
(d) $y = f(x_0) + f'(x_0)(x - x_0)$
(e) $y = f(x_0) - f'(x_0)(x - x_0)$

2 : The functions $f(x) = x^2$ and $g(x) = x^3$ are equal at $x = 0$ and at $x = 1$. Between $x = 0$ and at $x = 1$, for what value of x are their graphs furthest apart?
(a) $x = 1/2$ (b) $x = 2/3$ (c) $x = 1/3$ (d) $x = 1/4$ (e) $x = 3/4$

3 : Consider a point in the first quadrant on the hyperbola $x^2 - y^2 = 1$ with $x = 2$. The slope of the tangent line at that point is
(a) $2/\sqrt{3}$ (b) $2/\sqrt{5}$ (c) $1/\sqrt{3}$ (d) $\sqrt{5}/2$ (e) $2/3$

4 : For $a, b > 0$, solving the equation $\ln(x) = 2\ln(a) - 3\ln(b)$ for x leads to
(a) $x = e^{2a-3b}$ (b) $x = 2a - 3b$ (c) $x = a^2/b^3$ (d) $x = a^2b^3$ (e) $x = (a/b)^6$

5 : The function $y = f(x) = \arctan(x) - (x/2)$ has local maXima (LX), local minima (LM) and inflection points(IP) as follows:
(a) LX: $x = 1$, LM: $x = -1$, IP: $x = 0$.
(b) LX: $x = -1$, LM: $x = 1$, IP: $x = 0$.
(c) LX: $x = -1$, LM: $x = 1$, IP: none
(d) LX: $x = \sqrt{3}$, LM: $x = -\sqrt{3}$, IP: $x = 0$.
(e) LX: $x = -\sqrt{3}$, LM: $x = \sqrt{3}$, IP: $x = 0$.

0.2. **Related Rates:** Two spherical balloons are connected so that one inflates as the other deflates, the sum of their volumes remaining constant. When the first balloon has radius 10 cm and its radius is increasing at 3 cm/sec, the second balloon has radius 20 cm. What is the rate of change of the radius of the second balloon? [The volume of a sphere of radius r is $V = (4/3)\pi r^3$].

0.3. **Particle velocity:** A particle is moving along the x axis so that its distance from the origin at time t is given by

$$x(t) = (t + 2)^3 + \lambda t$$

where λ is a constant

(a) Determine the velocity $v(t)$ and the acceleration $a(t)$.

(b) Determine the minimum velocity over all time.

0.4. **Motion:** A particle's motion is described by $y(t) = t^3 - 6t^2 + 9t$
where $y(t)$ is the *displacement* (in metres) t is time (in seconds) and $0 \leq t \leq 4$ seconds.

(a) During this time interval, when is the particle furthest from its initial position ?

(b) During this time interval, what is the greatest speed of the particle?

(c) What is the total *distance* (including both forward and backward directions) that the particle has travelled during this time interval?

0.5. **Fish generations:** In Fish River, the number of salmon (in thousands), x, in a given year is linked to the number of salmon (in thousands), y, in the following year by the function

$$y = Axe^{-bx}$$

where $A, b > 0$ are constants.

(a) For what number of salmon is there no change in the number from one year to the next?

(b) Find the number of salmon that would yield the largest number of salmon in the following year.

0.6. **Blood alcohol:** Blood alcohol level (BAL), the amount of alcohol in your blood stream (here represented by $B(t)$, is measured in milligrams of alcohol per 10millilitres of blood. At the end of a party (time $t = 0$), a drinker is found to have $B(0) = 0.08$ (the legal level for driving impairment), and after that time, $B(t)$ satisfies the differential equation

$$\frac{dB}{dt} = -kB, \quad k > 0$$

where k is a constant that represents the rate of removal of alcohol form the blood stream by the liver.

(a) If the drinker had waited for 3 hrs before driving (until $= 3$), his BAL would have dropped to 0.04. Determine the value of the rate constant k (specifying appropriate units) for this drinker.

(b) According to the model, how much longer would it take for the BAL to drop to 0.01?

0.7. **Population with immigration:** An island has a bird population of density $P(t)$. New birds arrive continually with a constant colonization rate C birds per day. Each bird also has a constant probability per day, γ, of leaving the Island. At time $t = 0$ the bird population is $P(0) = P_0$

(a) Write down a differential equation that describes the rate of change of the bird population on the island.

(b) Find the steady state of that equation and interpret this in terms of the bird population.

c Write down the solution of the differential equation you found in (b) and show that it satisfies the following two properties: (i) the initial condition, (ii) as $t \to \infty$ it approaches the steady state you found in (b).

(d) If the island has no birds on it at time $t = 0$, how long would it take for the bird population to grow to 80% of the steady state value?

0.8. **Learning and forgetting:** Knowledge can be acquired by studying, but it is forgotten over time A simple model for learning represents the amount of knowledge, $y(t)$, that a person has at time t (in years) by a differential equation

$$\frac{dy}{dt} = S - fy$$

where $S \geq 0$ is the rate of studying and $f \geq 0$ is the rate of forgetting. We will assume that S and f are constants that are different for each person. [Your answers to the following questions will contain constants such as S or f.]

(a) Mary never forgets anything. What does this imply about the constants S and f? Mary starts studying in school at time $t = 0$ with no knowledge at all. How much knowledge will she have after 4 years (i.e. at $t = 4$)?

(b) Tom learned so much in preschool that his knowledge when entering school at time $t = 0$ is $y = 100$. However, once Tom in school, he stops studying completely. What does this imply about the constants S and f? How long will it take him to forget 75% of what he knew?

(c) Jane studies at the rate of 10 units per year and forgets at rate of 0.2 per year. Sketch a "direction field" ("slope field") for the differential equation describing Jane's knowledge. Add a few curves $y(t)$ to show how Jane's knowledge changes with time.

0.9. **Least cost:** A rectangular plot of land has dimensions L by D. A pipe is to be built joining points A and C. The pipe can be above ground along the border of the plot (Section AB), but has to be buried underground along the segment BC. The cost per unit length of the underground portion is 3 times that of the cost of the above ground portion. Determine the distance y so that the cost of the pipe will be as low as possible.

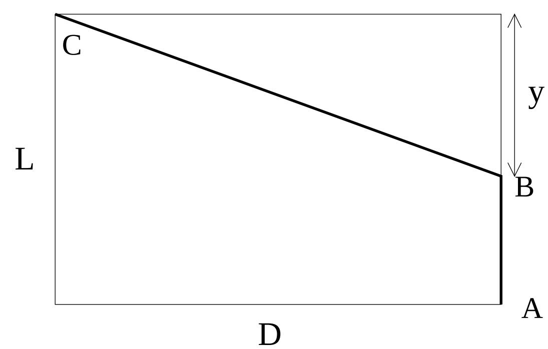

Figure 1. *Figure for problem 9*

0.10. **Ducks in a row:**
Graduate student Ryan Lukeman studies behaviour of duck flocks swimming near Canada Place in Vancouver, BC. This figure from his PhD thesis shows his photography set-up. Here $H = 10$ meters is the height from sea level up to his camera aperture at the observation point, $D = 2$ meters is the width of a pier (a stationary platform whose size is fixed), and x is the distance from the pier to the leading duck in the flock (in meters). α is a visual angle subtended at the camera, as shown. If the visual angle is increasing at the rate of 1/100 radians per second, at what rate is the distance x changing at the instant that $x = 3$ meters?

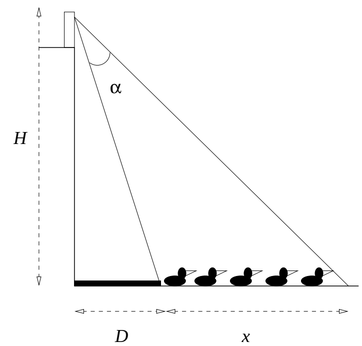

Figure 2. *Figure for problem 10*

0.11. **Circular race track:** Two runners are running around a circular race track whose length is 400m, as shown in Fig. 3(a). The first runner make a full revolution every 100s and the second runner every 150 s. They start at the same time at the start position, and the angles subtended by each runner with the radius of the start position are $\theta_1(t), \theta_2(t)$, respectively. As the runners go around the track both $\theta_1(t)$ and $\theta_2(t)$ will be changing with time.

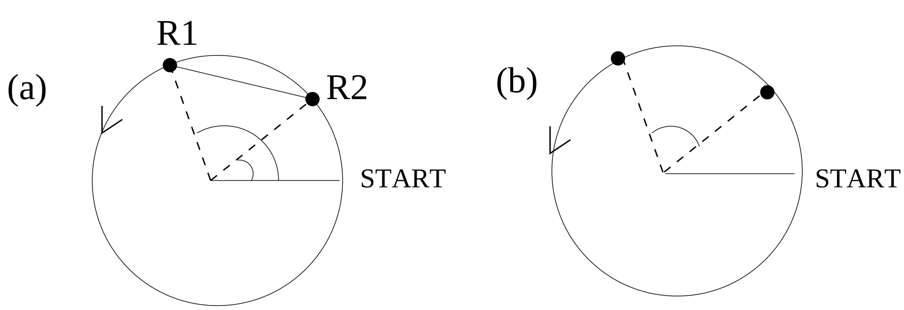

Figure 3. *Figure for problems 11 and 12. The angles in (a) are $\theta_1(t), \theta_2(t)$. In (b), the angle between the runners is ϕ.*

(a) At what rate it the angle $\phi = \theta_1 - \theta_2$ changing?

(b) What is the angle ϕ at $t = 25$s?

(c) What is the distance between the runners at $t = 25$s? (Here “distance” refers to the length of the straight line connecting the runners.)

(d) At what rate is the distance between the runners changing at $t = 25$s?

0.12. **Phase angle and synchrony:** Suppose that the same two runners as in Problem 11 would speed up or slow down depending on the angle between them, ϕ. (See Fig. 3). Then $\phi = \phi(t)$ will change with time. We will assume that the angle ϕ satisfies a differential equation of the form

$$\frac{d\phi}{dt} = A - B\sin(\phi)$$

where $A, B > 0$ are constants.

(a) What values of ϕ correspond to steady states (i.e. constant solutions) of this differential equation?

(b) What restriction should be placed on the constants A, B for these steady states to exist?

(c) Suppose $A = 1, B = 2$. Sketch the graph of $f(\phi) = A - B\sin(\phi)$ for $-\pi \leq \phi \leq \pi$ and use it to determine what will happen if the two runners start st the same point, ($\phi = 0$) at time $t = 0$.

0.13. **Logistic equation and its solution:**

(a) Show that the function

$$y(t) = \frac{1}{1 + e^{-t}}$$

satisfies the differential equation

$$\frac{dy}{dt} = y(1 - y).$$

(b) What is the initial value of y at $t = 0$?

(b) For what value of y is the growth rate largest?

(d) What will happen to y after a very long time?

0.0.1 Answers to Chapter 14 Problems

- **Problem 14.1:**
  1(d), 2(b), 3(a), 4(c) , 5(a)
- **Problem 14.2:**
  3/4 cm/sec.
- **Problem 14.3:**
  (a) $v(t) = 3(t+2)^2 + \lambda$, $a(t) = dv/dt = 6(t+2)$. (b) λ.
- **Problem 14.4:**
  (a) $t = 1$ and $t = 4$, (b) 9 m/s (c) 12 m.
- **Problem 14.5:**
  (a)$x = 0, \ln(A)/b$ (b)$x = 1/b$.
- **Problem 14.6:**
  (a) $k = \ln(2)/3$ per hr (b) 6 more hrs.
- **Problem 14.7:**
  (a) $dP/dt = C - \gamma P$ (b) $P = C/\gamma$ (d) $t = (\ln(1/0.2)/\gamma$.
- **Problem 14.8:**
  (a) $y = 4S$ (b) $T = 2\tau_{1/2} = 2\ln(2)/f$ (c) $y \to 50$
- **Problem 14.9:**
  $y = D/\sqrt{8}$
- **Problem 14.10:**
  $0.125 \mathrm{m/s}$
- **Problem 14.11:**
  (a) $\pi/150$ radians/s (b) $\pi/6$ radians (c) $D = \frac{200}{\pi}(2-\sqrt{3})^{1/2}$ (d) $\frac{dD}{dt} = \frac{2}{3(2-\sqrt{3})^{1/2}}$
- **Problem 14.12:**
  (a) $\phi = \arcsin(A/B)$, (b) $-1 < \arcsin(A/B) < 1$, (c) $\phi \to \pi/6$
- **Problem 14.13:**
  (b) $y(0) = 0.5$ (c) at $y = 0.5$ (d) $y \to 1$.