Mathematics 102 Review Questions

Problem 1: Multiple Choice Questions

1: Consider the function $y = f(x) = 3e^{-2x} - 5e^{-4x}$

(a) The function has a local maximum at $x = (1/2)\ln(10/3)$
(b) The function has a local minimum at $x = (1/2)\ln(10/3)$
(c) The function has a local maximum at $x = (-1/2)\ln(3/5)$
(d) The function has a local minimum at $x = (1/2)\ln(3/5)$
(e) The function has a local maximum at $x = (-1/2)\ln(3/20)$

2: Let m_1 be the slope of the function $y = 3^x$ at the point $x = 0$ and let m_2 be the slope of the function $y = \log_3 x$ at $x = 1$ Then

(a) $m_1 = \ln(3)m_2$ (b) $m_1 = m_2$ (c) $m_1 = -m_2$ (d) $m_1 = 1/m_2$ (e) $m_1 = m_2/\ln(3)$

3: Consider the curve whose equation is $x^4 + y^4 + 3xy = 5$. The slope of the tangent line, dy/dx, at the point $(1, 1)$ is

(a) 1 (b) -1 (c) 0 (d) -4/7 (e) 1/7

4: Two kinds of bacteria are found in a sample of tainted food. It is found that the population size of type 1, N_1 and of type 2, N_2 satisfy the equations

$$\frac{dN_1}{dt} = -0.2N_1, \quad N_1(0) = 1000, \qquad \frac{dN_2}{dt} = 0.8N_2, \quad N_2(0) = 10.$$

Then the population sizes are equal $N_1 = N_2$ at the following time:

(a) $t = \ln(40)$ (b) $t = \ln(60)$ (c) $t = \ln(80)$ (d) $t = \ln(90)$ (e) $t = \ln(100)$

5: In a conical pile of sand the ratio of the height to the base radius is always $r/h = 3$. (Recall that the volume of a cone with height h and radius r is $V = (\pi/3)r^2h$.) If the volume is increasing at rate 3 m^3/min, how fast (in m/min) is the height changing when $h = 2$m?

(a) $1/(12\pi)$ (b) $(1/\pi)^{1/3}$ (c) $27/(4\pi)$ (d) $1/(4\pi)$ (e) $1/(36\pi)$

6: Newton's Law of cooling leads to a differential equation that predicts the temperature $T(t)$ of an object whose initial temperature is T_0 in an environment whose temperature is E. The predicted temperature is given by $T(t) = E + (T_0 - E)e^{-kt}$ where t is time and k is a constant. Shown in Fig 1 on the following page is some data points plotted as $\ln(T(t) - E)$ versus time in minutes. The ambient temperature was $E = 22^\circ$ C. Also shown on the graph is the line that best fits those 11 points. According to this graph, the value of the constant k is approximately

(a) -1/27 (b) $e^{1/27}$ (c) 1/27 (d) 4/27 (e) $\ln(1/27)$

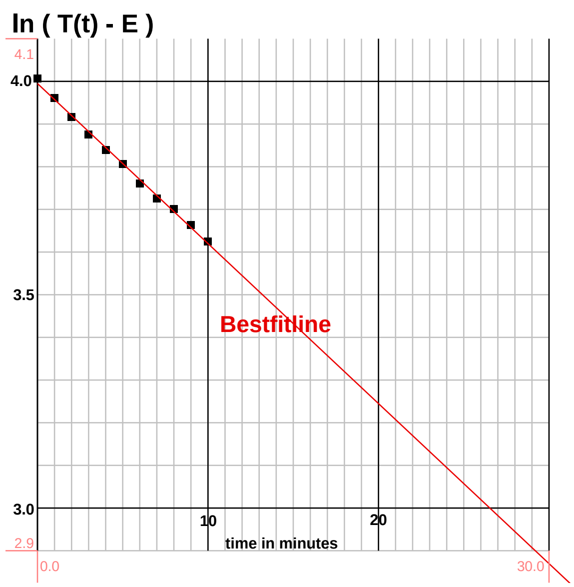


Figure 1: Figure for Multiple Choice problem 6

Long Answer Problems

Problem 2a: Fig. 2 shows a 1 km race track with circular ends. Find the values of x and y that will maximize the area of the rectangle.

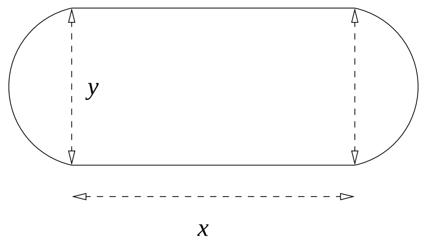


Figure 2: This shape is investigated in both problems 1 and 2.

Problem 2b: Now suppose that Fig 2 shows the shape of a leaf of some plant. If the plant grows so that x increases at the rate 2 cm/year and y increases at the rate 1 cm/year, at what rate will the leaf's entire area be increasing?

Problem 3: Find the dimensions of the largest rectangle that can fit exactly into a circle whose radius is 10 cm.

Problem 4: A cell of the bacterium E.coli has the shape of a cylinder with two hemispherical

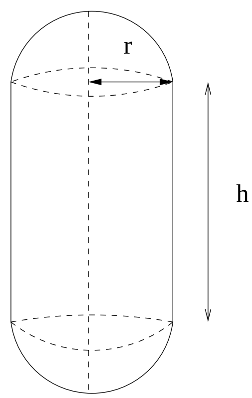


Figure 3: Shape of the object described in Problem 4. Note: Useful volumes and surface areas: For a hemisphere, $V = (2/3)\pi r^3, S = 2\pi r^2$. For a cylinder, $V = \pi r^2 h$ and $S = 2\pi r h$ (not including end caps)

caps, as shown in Fig 3. Consider this shape, with h the height of the cylinder, and r the radius of the cylinder and hemispheres. (a) Find the values of r and h that lead to the largest volume for a fixed constant surface area, S= constant. (b) Describe or sketch the shape you found in (a). (c) A typical E. coli cell has $h = 1\mu$m and $r = 0.5\mu$m. Based on your results in (a) and (b), would you agree that E. coli has a shape that maximizes its volume for a fixed surface area? (Explain your answer).

Problem 5: If the cell shown above in Fig 3 is growing so that the height increases twice as fast as the radius. If the radius is growing at 1 μm per day at what rate will the volume of the cell increase? (Leave your answer in terms of the height and radius of the cell.)

Problem 6:(a) It takes you 1 hrs (total) to travel to and from UBC every day to study Philosophy 101. The amount of new learning (in arbitrary units) that you can get by spending t hours at the university is given approximately by

$$L_P(t) = \frac{10t}{9+t}.$$

How long should you stay at UBC on a given day if you want to maximize your learning per time spent? (Time spent includes travel time.) (b) If you take Math 10000 instead of Philosophy, your learning at time t is

$$L_M(t) = t^2.$$

How long should you stay at UBC to maximize your learning in that case?

Problem 7: The atoms of some radioactive material are known to have a probability k of decaying per unit time. We will use $y(t)$ to denote the amount of radioactivity remaining at time t. Suppose that there is 100 gm of this radioactive substance initially, at time $t = 0$. Consider what happens during a small time interval Δt. How much radioactive material is left at time $t_1 = \Delta t$? At time $t_s = 2\Delta t$? Write down an equation that links $y(t_{n+1})$ to $y(t_n)$ where $t_n = n\Delta t$. Convert this equation to a differential equation. If the half life of this substance is 1 day, find out how much is left after 5 days. What is the value of k in this case, and how much radioactivity is left at time t?

Problem 8: Given a population of 6 billion people on Planet Earth, and using the approximate growth rate of $r = 0.0125$ per year, how long ago was this population only 1 million? Assume that the growth has been the same throughout history (which is not actually true).

Problem 9: Find critical points for the function $y = e^x(1 - \ln(x))$ for $0.1 \leq x \leq 2$ and classify their types.

Problem 10: The function $y = \ln(x) - e^x$ has a critical point in the interval $0.1 \leq x \leq 2$. It is not possible to solve for the value of x at that point, but it is possible to find out what kind of critical point that is. Determine whether that point is a local maximum, minimum, or inflection point.

Problem 11: (a) Consider the polynomial $y = 4x^5 - 15x^4$. Find all local minima maxima, and inflection points for this function.

(b) Find the global minimum and maximum for the function in problem (1) on the interval [-1,1].

Problem 12: Consider the polynomial $y = -x^5 - x^4 + 3x^3$. Use calculus to find all local minima maxima, and inflection points for this function.

(b) Find the global minimum and maximum for the function in Problem (3) on the interval [-1,1].

Problem 13: Find a polynomial of third degree that has a local maximum at $x = 1$, a zero and an inflection point at $x = 0$, and goes through the point (1,2). Hint: assume $p(x) = ax^3 + bx^2 + cx + d$ and find the values of a, b, c, d.

Problem 14: Find a linear approximation to the function $y = x^2$ at the point whose x coordinate is $x = 2$. Use your result to approximate the value of $(2.0001)^2$.

Problem 15: The Lennard-Jones potential, $V(x)$ is the potential energy associated with two uncharged molecules a distance x apart, and is given by the formula

$$V(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

Molecules would tend to adjust their separation distance so as to minimize this potential. Find any local maxima or minima of this potential. Find the distance between the molecules, x, at which $V(x)$ is minimized and use the second derivative test to verify that this is a local minimum.

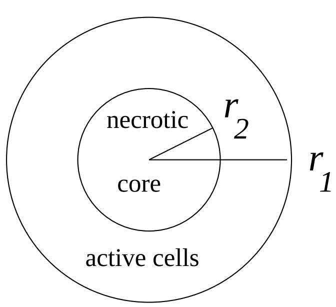
Problem 16: Consider an object thrown upwards with initial velocity $v_0 > 0$ and initial height $h_0 > 0$. Then the height of the object at time t is given by

$$y = f(t) = -\frac{1}{2}gt^2 + v_0 t + h_0.$$

Find critical points of $f(t)$ and use both the second and first derivative tests to establish that this is a local maximum.

Problem 17: The figure (not drawn to scale) shows a tumor mass containing a necrotic (dead) core (radius r_2), surrounded by a layer of actively dividing tumor cells. The entire tumor can be assumed to be spherical, and the core is also spherical[1].

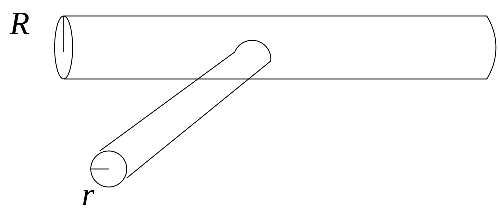
(a) If the necrotic core increases at the rate 3 cm^3/year and the volume of the active cells increases by 4 cm^3/year, at what rate is the outer radius of the tumor (r_1) changing when $r_1 = 1$ cm. (NOTE: Show all your work, leave your answer as a fraction in terms of π; indicate units with your answer.)

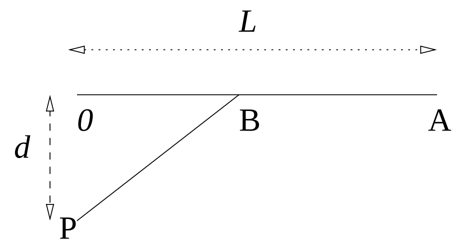


(b) At what rate (in cm^2/yr) does the outer surface area of the tumor increase when $r_1 = 1$cm?

Problem 18: Shown below is a major artery, (radius R) and one of its branches (radius r). A labeled schematic diagram is also shown (right). The length 0A is L, and the distance between 0 and P is d, where 0P is perpendicular to 0A. The location of the branch point (B) is to be determined so that the total resistance to blood flow in the path ABP is as small as possible. (R, r, d, L are positive constants, and $R > r$.)

[1] Recall that the volume and surface area of a sphere are $V = (4/3)\pi r^3$, $S = 4\pi r^2$





(a) Let the distance between 0 and B be x. What is the length of the segment BA and what is the length of the segment BP?

(b) *The resistance of any blood vessel is proportional to its length and inversely proportional to its radius to the fourth power*[2]. Based on this fact, what is the resistance, T_1, of segment BA and what is the resistance, T_2, of the segment BP?

(c) Find the value of the variable x for which the total resistance, $T(x) = T_1 + T_2$ is a minimum.

[2] "z is inversely proportional to y" means that $z = k/y$ for some constant k