Chapter 5
Tangent lines, linear approximation, and Newton’s method

In a previous chapter, we defined the **tangent line** as the line we see when we zoom into the graph of a (continuous) function \( y = f(x) \) at some point. In much the same sense, the tangent line approximate the local behaviour of a function near the **point of tangency**. Given the point of tangency \( x_0, y_0 = f(x_0) \), and the slope \( m = f'(x_0) \) (the derivative), we can find the equation of the tangent line

\[
\frac{\text{rise}}{\text{run}} = \frac{y - y_0}{x - x_0} = m = f'(x_0) \quad \Rightarrow \quad y = f(x_0) + f'(x_0)(x - x_0). \quad (5.1)
\]

(See Appendix A for a review of straight lines.) We use Eqn. (5.1) in several important applications, including **linear approximation**, a method for estimating the value of a function near the point of tangency. A further application of the tangent line is to **Newton’s method** for locating zeros of a function (values of \( x \) for which \( f(x) = 0 \)).

### 5.1 The equation of a tangent line

**Section 5.1 Learning goals**

1. Given a simple function \( y = f(x) \) and a point \( x \), be able to find the equation of the tangent line to the graph at that point.

2. Be able to graph both the function and its tangent line using a spreadsheet or your favorite software.

In the following examples, the equation of the tangent line is easily found.

**Example 5.1 (Tangent to a parabola)** Find the equations of the tangent lines to the parabola \( y = f(x) = x^2 \) at the points \( x = 1 \) and \( x = 2 \) (“Line 1” and “Line 2”). Then determine whether these tangent lines intersect, and if so, where.
Solution: The slopes of a tangent line is a derivative, which in this case is \( f'(x) = 2x \). So \( m_1 = f'(1) = 2 \cdot 1 = 2 \) (for Line 1) and \( m_2 = f'(2) = 2 \cdot 2 = 4 \) (for Line 2). The points of tangency are on the curve, \((x, x^2)\), so these are \((1,1)\) for Line 1 and \((2,4)\) for Line 2. Thus with slope and point for each line, we find that

Line 1: \[
\frac{y - 1}{x - 1} = m_1 = 2, \quad \Rightarrow \quad y = 1 + 2(x - 1) \quad \Rightarrow \quad y = 2x - 1,
\]

Line 2: \[
\frac{y - 4}{x - 2} = m_2 = 4 \quad \Rightarrow \quad y = 4 + 4(x - 2) \quad \Rightarrow \quad y = 4x - 4.
\]

Two lines intersect when their y values (and x values) are the same. Solving for x we get

\[
2x - 1 = 4x - 4 \quad \Rightarrow \quad -2x = -3 \quad \Rightarrow \quad x = \frac{3}{2}.
\]

so indeed the two tangent lines intersect at \( x = 3/2 \) as shown in Fig. 5.1.

Figure 5.1. The graph of the parabola \( y = f(x) = x^2 \) and its tangent lines at \( x = 1 \) and \( x = 2 \). See Example 5.1 for the equations and point of intersection of these tangent lines.

The next example points to the fact that a tangent line can be used to approximate the zero of a function. This idea will be developed into a useful approximation method called Newton’s method.

Example 5.2 Draw the graph of the function \( y = f(x) = x^3 - x \) together with its tangent line at the point \( x = 1.5 \). Where does that tangent line intersect the x axis? Compare that point of intersection with a zero of the function.

Solution: The coordinates of the point of interest \((x, f(x))\) are \((1.5, f(1.5)) = (1.5, 1.875)\). The derivative of \( f(x) = x^3 - x \) is \( f'(x) = 3x^2 - 1 \). A tangent line at the point \( x = 1.5 \) has slope \( m = f'(1.5) = 3(1.5)^2 - 1 = 5.75 \). Thus, the equation of the tangent line is

\[
\frac{y - 1.875}{x - 1.5} = 5.75 \quad \Rightarrow \quad y = 1.875 + 5.75(x - 1.5) \quad \Rightarrow \quad y = 5.75x - 6.75.
\]
5.1. The equation of a tangent line

Figure 5.2. The graph of the function \( y = f(x) = x^3 - x \) is shown in black, together with its tangent line at the point \( x = 1.5 \). In this low magnification view, we see that the tangent line stays close to the graph of the function only close to the point of tangency. Away from that point, it strays off.

The tangent line intersects the \( x \) axis when \( y = 0 \), which occurs at

\[
0 = 5.75x - 6.75 \quad \Rightarrow \quad x = \frac{6.75}{5.75} = 1.174.
\]

This is close to\(^{22}\) a zero of \( f(x) \). We show the graph of the function \( f(x) = x^3 - x \) and its tangent line, \( y = 5.75x - 6.75 \) in Fig. 5.2.

We will shortly discuss how a repetition of the same idea can be used to refine the approximation of a zero using Newton’s method.

**Example 5.3** (a) Find the equation of the tangent line to \( y = f(x) = x^3 - ax \) for \( a > 0 \) constant, at the point \( x = 1 \). (b) Find where that tangent line intersects the \( x \) axis.

**Solution:** This is the same type of calculation, but the constant, \( a \), makes the example a little less straightforward. (a) The derivative of \( f(x) = x^3 - ax \) is \( f'(x) = 3x^2 - a \) so at \( x = 1 \), the slope is \( m = f'(1) = 3 - a \). The point of tangency is \((1, f(1)) = (1, 1 - a)\). Then, the equation of the tangent line is

\[
y - (1 - a) = (3 - a)(x - 1) \quad \Rightarrow \quad y = (3 - a)(x - 1) + (1 - a) \quad \Rightarrow \quad y = (3 - a)x - 2.
\]

[Remark: at this point is is wise to check that the tangent line goes through the desired point and has the slope we found. One way to do this is to pick a simple value for \( a \), e.g. \( a = 1 \) and do a quick check that the answer matches what we have found.]

(b) To find the point of intersection, set \( y = (3 - a)x - 2 = 0 \) and solve for \( x \) to obtain \( x = 2/(3 - a) \).

\(^{22}\)In this example, we could directly solve \( f(x) = x^3 - x = 0 \) to obtain \( x = 0, \pm 1 \). But in many cases that is not possible, as we will soon see.
Example 5.4 Find the equation of the tangent line to the function $y = f(x) = \sqrt{x}$ at the point $x = 4$.

Solution: By (3.6), the derivative of $y = f(x) = \sqrt{x}$ is $f'(x) = 1/(2\sqrt{x})$. At $x = 4$, the slope is $f'(4) = 1/(2\sqrt{4}) = 1/4$ and the point of tangency is $(4, 2)$. Then the equation of the tangent line is

$$\frac{y - 2}{x - 4} = 0.25 \Rightarrow y = 2 + 0.25(x - 4).$$

5.2 Generic tangent line equation and properties

Section 5.2 Learning goals

1. Understand the generic form of the tangent line equation, (5.1) and be able to connect it to the geometry of the tangent line.

2. Be able to find the coordinate of the point at which the tangent line intersects the $x$ axis (important for Newton’s Method later on in Section 5.4).

5.2.1 Generic tangent line equation

We motivate the general equation of a tangent line (5.1) to an arbitrary function $f(x)$ at a point of tangency $x_0$. Shown in Fig. 5.3 is a continuous function $y = f(x)$, assumed to be differentiable at some point $x_0$ at which a tangent line is attached. We see that: (1) The line goes through the point $(x_0, f(x_0))$. (2) The line has slope given by the derivative of the function at the point of interest, that is, $m = f'(x_0)$. Then from the slope-point form of the equation of a straight line,

$$\frac{y - f(x_0)}{x - x_0} = m = f'(x_0).$$

Figure 5.3. The graph of an arbitrary function $y = f(x)$ and a tangent line at $x = x_0$. The equation of this generic tangent line is (5.2).
Rearranging this and eliminating the notation \( m \), we have the desired result, repeated here for convenience:

\[
y = f(x_0) + f'(x_0)(x - x_0).
\]  

(5.2)

### 5.2.2 Where a tangent line intersects the \( x \) axis

We ask where the generic tangent line equation (5.2) intersects the \( x \) axis. The result will prove useful shortly.

**Example 5.5** Let \( y = f(x) \) be a smooth function, differentiable at \( x_0 \), and suppose that (5.2) is the equation of the tangent line to the curve at \( x_0 \). Find the coordinate of the point at which this tangent line intersects the \( x \) axis.

**Solution:** At the intersection with the \( x \) axis, we have \( y = 0 \). Plugging this into (5.1) leads to

\[
0 = f(x_0) + f'(x_0)(x - x_0) \quad \Rightarrow \quad (x - x_0) = -\frac{f(x_0)}{f'(x_0)} \quad \Rightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.
\]

Thus the desired \( x \) coordinate, which we will refer to as \( x_1 \) is

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.
\]  

(5.3)

This result will turn out to be of particular relevance in Section 5.4, where we discuss Newton’s method for approximating the zeros of a function.

### 5.3 Approximating a function by its tangent line

**Section 5.3 Learning goals**

1. Understand that a tangent line approximates the behaviour of a function close to the point of tangency.

2. Be able to use this idea to find a linear approximation to a value of a given function at some point.

3. Be able to determine whether the linear approximation overestimates or underestimates the value of the function.

We have already encountered the idea that the tangent line approximates the local behaviour of a function, at least close enough to the point of tangency. Here we utilize this idea in a formal procedure called linear approximation. The idea is to choose a point (often called the base point) at which the value of the function and of its derivative are known, or are easy to calculate and use the tangent line at that point to estimate values of the function in the vicinity. Specifically,
1. The generic equation of the tangent line to \( y = f(x) \) at \( x_0 \) is given by Eqn. (5.2). That line approximates the behaviour of the function close to \( x_0 \), and leads to the so-called linear approximation of the function:

\[
y = f(x_0) + f'(x_0)(x - x_0) \approx f(x) \quad \Rightarrow \quad f(x) \approx f(x_0) + f'(x_0)(x - x_0).
\]

2. The approximation is exact at \( x = x_0 \), and holds well provided \( x \) is close to \( x_0 \). (The expression on the right hand side is precisely the value of \( y \) on the tangent line at \( x = x_0 \).

**Example 5.6** Use the fact that the derivative of the function \( f(x) = x^2 \) is \( f'(x) = 2x \) (as found in Example 2.21) to find a linear approximation for the value \((10.03)^2\).

**Solution:** It is easy to compute the value of \( f(x) \) at the nearby point \( x = 10 \) since \( 10^2 = 100 \). From the derivative \( f'(x) = 2x \), the slope of the tangent line at \( x = 10 \) is \( f'(10) = 2(10) = 20 \). The equation of the tangent line, and linear approximation of the function are:

\[
\frac{y - 100}{x - 10} = 20 \quad \Rightarrow \quad y = 100 + 20(x - 10), \quad \Rightarrow \quad f(x) \approx 100 + 20(x - 10).
\]

On the tangent line, the value of \( y \) corresponding to \( x = 10.03 \) which is our approximation to the value of the function is

\[
f(10.03) \approx y = 100 + 20(10.03 - 10) = 100 + 20(0.03) = 100.6
\]

This compares well with the true value of \( f(10.03) = 100.6009 \) found using a calculator for the actual function.

**Figure 5.4.** Functions (black curves) and their linear approximations (red) for Examples 5.6 and 5.7. Whenever the tangent line is below (above) the curve, we say that the linear approximation under (over)-estimates the value of the function.
Example 5.7 (Approximating the sine of a small angle) Use a linear approximation to find a rough value for \( \sin(0.1) \).

Solution: Close to \( x = 0 \) the function \( y = \sin(x) \) is well approximated by its tangent line, \( y = x \) (see Example 3.4). Hence, the linear approximation of \( y = \sin(x) \) near \( x = 0 \) is \( f(x) \approx y = x \) (provided \( x \) is in radians, as discussed in Chapter 14). Thus, at \( x = 0.1 \) radians, we find that \( \sin(0.1) = 0.09983 \approx 0.1 \).

5.3.1 Accuracy of the linear approximation

Example 5.8 (Over or underestimate?) Determine in each of the previous examples whether the linear approximation over or underestimates the true value of the function.

Solution: We show the functions and their linear approximations in Fig. 5.4(a,b). In (a) we find that the tangent line to \( y = x^2 \) is always underneath the graph of the function, so that a linear approximation underestimates the true value of the function. In (b), we see that the tangent line to \( y = \sin(x) \) at \( x = 0 \) is above the graph for \( x > 0 \) and below the graph for \( x < 0 \). This meant that the linear approximation is larger than (overestimates) the function for \( x > 0 \) and smaller than (underestimates) the function for \( x < 0 \). Later, we will associate these properties with the concavity of the function, that is, whether the graph is locally concave up or down.

Example 5.9 Use linear approximation to estimate the value of \( \sqrt{6} \). Then determine whether the linear approximation under-estimates or over-estimates the function.

Figure 5.5. Linear approximation based at \( x = 4 \) to the function \( y = f(x) = \sqrt{x} \).
Solution: The derivative of \( y = f(x) = \sqrt{x} = x^{1/2} \) is \( f'(x) = 1/(2\sqrt{x}) = (1/2)x^{-1/2} \).

Both the function and its derivative require evaluation of a square root. Some numbers (perfect squares\(^{23}\)) have convenient square roots. \( x = 4 \), is such a number nearby, so we use it as the “base point” for a linear approximation. The slope of the tangent line at \( x = 4 \) is \( f'(4) = 1/(2\sqrt{4}) = 1/4 = 0.25 \) so the linear approximation of \( \sqrt{6} \) is obtained as

\[
y = f(4) + f'(4)(x - 4) \quad \Rightarrow \quad y = 2 + 0.25(x - 4) \quad \Rightarrow \quad \sqrt{6} \approx 2 + 0.25(6 - 4) = 2.5.
\]

A graph of the function and its tangent line in Fig. 5.5(a) and a zoomed portion in Fig. 5.5(b) compares the true\(^{24}\) and approximated values of \( \sqrt{6} \). The tangent line is above the graph of the function, so the linear approximation overestimates values of the function. The discrepancy between true and approximated values is called the error. The closer we are to the base point, the smaller the error in the approximation. This is also demonstrated by comparing the values in Table 5.1, computed using a spreadsheet.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} ) (exact value)</th>
<th>( y = f(x_0) + f'(x_0)(x - x_0) ) (approx value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>2.0000</td>
<td>1.4142</td>
<td>1.5000</td>
</tr>
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<td>4.0000</td>
<td>2.0000</td>
<td>2.0000</td>
</tr>
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<td>6.0000</td>
<td>2.4495</td>
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</tr>
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</tr>
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<td>3.1623</td>
<td>3.5000</td>
</tr>
<tr>
<td>12.0000</td>
<td>3.4641</td>
<td>4.0000</td>
</tr>
<tr>
<td>14.0000</td>
<td>3.7417</td>
<td>4.5000</td>
</tr>
<tr>
<td>16.0000</td>
<td>4.0000</td>
<td>5.0000</td>
</tr>
</tbody>
</table>

Table 5.1. Linear approximation to \( \sqrt{x} \). The exact value is recorded in column 2 and the linear approximation in column 3.

5.4 Tangent lines for finding zeros of a function

Section 5.4 Learning goals

1. Understand the geometry on which Newton’s method is based (Fig. 5.6).

2. Given \( f(x) \) and initial guess \( x_0 \), be able to use Newton’s method to find improved values \( x_1, x_2, \) etc., for the zero of \( f(x) \) (value of \( x \) such that \( f(x) = 0 \)).

3. Be able to select a suitable initial guess \( x_0 \).

\(^{23}\)A perfect square is an integer of the form \( m = n^2 \) where \( n \) is also an integer.

\(^{24}\)The actual value, computed on a calculator is \( \sqrt{6} = 2.449.. \)
5.4. Tangent lines for finding zeros of a function

**Definition 5.10 (Zero).** Given a function $y = f(x)$, we say that $x^*$ is a **zero** of $f$ if $f(x^*) = 0$. In this case we also say that “$x^*$ is a **root** of the equation $f(x) = 0$”.

In many cases, it is impossible to compute a value of a zero, $x^*$ analytically. Based on tangent line approximations, we next discuss **Newton's method**, an approximation that does the job!

![Figure 5.6](image.png)

**Figure 5.6.** In Newton’s method, we seek a decimal approximation for $x^*$, a zero of $y = f(x)$. A rough initial guess $x_0$ is refined by “sliding down the tangent line” (glued to the curve at $x_0$). This brings us to an improved guess $x_1$. Repeating this again and again allows us to find the root to any desired accuracy.

### 5.4.1 Newton’s method

Consider the function $y = f(x)$ shown in Figure 5.6. We want to find a decimal approximation for the value $x$ such that $f(x) = 0$. In the figure, this value is denoted by $x^*$. Newton’s method is an **iterated scheme**, that can be applied multiple times, to generate a decimal expansion of the desired zero to any level of accuracy.

To apply the method, we require a starting value, $x_0$, that is a very rough initial guess for the desired root. (How to find this initial guess will be discussed later.) Newton’s method is a recipe for getting better and better approximations of the true value, $x^*$.

The tangent line to the graph of $f(x)$ at $x_0$ approximates the behaviour of the function. Hence, we hope that the point at which the tangent line intersects the $x$ axis (denoted $x_1$) is an approximation of the zero of the function. Usually it is true that $x_1$ is closer to $x^*$ than the initial guess$^{25}$. Now use $x_1$ as the (improved) guess, and repeat the process. The values $x_2, x_3$ will rapidly approach the desired root $x^*$. From (5.3), we already found that the “formula” for $x_1$ is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

---

$^{25}$There are some cases where this fails to be true, and we discuss what can go wrong shortly
Repeating the procedure generates the values

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]

\[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \]

\[ x_4 = \ldots \]

In practice, when it works, Newton’s method converges quite rapidly, that is, it approaches the root with excellent accuracy, after very few repetitions (iterations). To summarize,

**Newton’s method**: Given an approximation \( x_k \) for the root of the equation \( f(x) = 0 \), we can improve the accuracy of that approximation with another iteration using

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

**Example 5.11** Find zeros of the function \( y = f(x) = x^3 - x - 3 \) starting with the initial guess \( x_0 = 1 \).

**Solution**: The derivative is \( f'(x) = 3x^2 - 1 \), so Newton’s method recipe for the improved guess is

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - x_0 - 3}{3x_0^2 - 1} \]

Starting with \( x_0 = 1 \), we obtain

\[ x_1 = 1.727272727, x_2 = 1.673691174, x_3 = 1.67170257, x_4 = 1.671699882. \]

The iterates converge to the result \( x \approx 1.6717 \).

**Example 5.12** Use Newton’s method to find a decimal approximation of the square root of 6.

**Solution**: It is first necessary to restate the problem in the form “Find a value of \( x \) such that a certain function \( f(x) = 0 \).” Clearly, one function that would accomplish this\(^2\) is

\[ f(x) = x^2 - 6 \]

since \( f(x) = 0 \) corresponds to \( x^2 - 6 = 0 \), i.e. \( x = \sqrt{6} \). Then the derivative of \( f(x) \) is \( f'(x) = 2x \), and the recipe to repeat is

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 6}{2x_0}. \]

\(^2\)We could also find other functions that have the same property, e.g. \( f(x) = x^4 - 36 \), but the above is one of the simplest such functions.
5.4. Tangent lines for finding zeros of a function

Newton’s method

\[ f(x) = x^2 - 6 \]

Table 5.2. Newton’s method applied to Example 5.12. We start with \( x_0 = 1 \) as our initial approximation and refine it four times.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x_k )</th>
<th>( f(x_k) )</th>
<th>( f'(x_k) )</th>
<th>( x_{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
<td>-5.00</td>
<td>2.00</td>
<td>3.5</td>
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<td>1</td>
<td>3.5</td>
<td>6.250</td>
<td>7.00</td>
<td>2.6071</td>
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<tr>
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<td>2.6071</td>
<td>0.7972</td>
<td>5.2143</td>
<td>2.4543</td>
</tr>
<tr>
<td>3</td>
<td>2.4543</td>
<td>0.0234</td>
<td>4.9085</td>
<td>2.4495</td>
</tr>
<tr>
<td>4</td>
<td>2.4495</td>
<td>0.000</td>
<td>4.8990</td>
<td>2.4495</td>
</tr>
</tbody>
</table>

Figure 5.7. Newton’s method applied to solving \( y = f(x) = x^2 - 6 = 0 \).

Starting with an initial guess \( x_0 = 1 \) (not very close to the value of the root), we show in Figure 5.7(a) how Newton’s method applies a tangent line to determination \( x_1 \). In Fig. 5.7(b), we see how the value of \( x_1 \) is then used to obtain \( x_2 \) by repeating the calculation.

A spreadsheet is ideal carrying out the repetitive calculations, as shown in Table 5.2. For example, we compute the following set of values using our spreadsheet. Observe that the fourth column contains the computed (Newton’s method) values, \( x_1, x_2, \) etc. These values are then copied onto the first column to be used as new “initial guesses”. After several repetitions, the numbers calculated converge to 2.4495, and no longer change to that level of accuracy displayed. This signals that we have obtained to root to 5 significant figures of accuracy.
5.5 Harder problems: Finding the point of tangency

In this section, we present a sample of problems in which the path to a solution is more subtle. In some of these, finding the point of tangency is part of the question. We must use clues about the function to solve for that point, as well as construct the tangent line equation from information supplied. In other cases, the problem involves a parameter whose value is not specified initially. Such examples are meant to hone problem solving skills.

Example 5.13 Find any value(s) of the constant $a$ such that the line $y = ax$ is tangent to the curve $y = f(x) = -x^2 + 3x - 2$.

Solution: We do not know the coordinate of any such point, but we will label it $x_0$ in Figure 5.8 to denote that it is a definite (as yet to be determined) value. Finding $x_0$, will be part of the problem. We collect the information to be used:

- The tangent line $y = ax$ intersects the graph of the function $y = f(x)$ at $x_0$.
- The equation of the tangent line is $y = ax$, so its slope, which is the derivative of $f(x)$ at $x_0$ is $a$.

Using these two facts, we can write down the following equations:

- Equating slopes:
  \[
  f'(x_0) = -2x_0 + 3 = a.
  \]
- Equating $y$ values on line and graph of $f(x)$:
  \[
  f(x_0) = -x_0^2 + 3x_0 - 2 = ax_0.
  \]

These are two equations for two unknowns, $(a$ and $x_0$). We can solve this system easily by substituting the value of $a$ from the first equation into the second, getting

\[
-x_0^2 + 3x_0 - 2 = (-2x_0 + 3)x_0.
\]
5.5. Harder problems: Finding the point of tangency

Simplifying:
\[-x_0^2 + 3x_0 - 2 = -2x_0^2 + 3x_0 \quad \Rightarrow \quad x_0^2 - 2 = 0, \quad x_0 = \pm \sqrt{2}.
\]

Thus, there are two possible points of tangency, as shown in Figure 5.9. Finally, we find \( a \) using \( a = -2x_0 + 3 \). We get:
\[ x_0 = \sqrt{2} \quad \Rightarrow \quad a = -2\sqrt{2} + 3, \quad \text{and} \quad x_0 = -\sqrt{2} \quad \Rightarrow \quad a = 2\sqrt{2} + 3. \]

We have seen that the solution was set up as a list of information provided, a set of equations based on that information, and a chain of reasoning to arrive at the final solution. Practicing such multi-step problems is an important part of training for science, medicine, engineering, and other fields.

Example 5.14 Find the equation of the tangent line to the curve \( y = f(x) = 1 - x^2 \) that goes through the point (1,1).

Solution: Finding the point of tangency \( x_0 \) is part of the problem. We use the following facts: (1) The tangent line goes through the point \((x_0, f(x_0))\) on the graph of the function and has slope \( f'(x_0) \). (2) Consequently, its equation will have the form (5.2). For the given function and point of tangency \( x_0 \), we have
\[ f(x_0) = 1 - x_0^2, \quad f'(x_0) = -2x_0. \]

Hence the tangent line equation is
\[ y = f(x_0) + f'(x_0)(x - x_0) = (1 - x_0^2) - 2x_0(x - x_0). \]

We are told that this line goes through the point \((x, y) = (1, 1)\) so that
\[ 1 = (1 - x_0^2) - 2x_0(1 - x_0), \quad \Rightarrow \quad 0 = x_0^2 - 2x_0, \quad \Rightarrow \quad x_0 = 2x_0. \]

Thus, there are two possible points of tangency, \( x_0 = 0, 2 \) and two tangent lines that satisfy the given condition. Plugging in these two values of \( x_0 \) into the generic equation for \( y \) leads to the two tangent line equations
\[ y = 1, \quad \text{and} \quad y = (1 - 2^2) - 2 \cdot 2(x - 2) = -3 - 4(x - 2). \]

It is easily checked that both lines go through the point (1,1) as required.
Example 5.15  Shown in Fig. 5.10 is the function
\[ f(x) = C \frac{x}{x + a} \]
together with one of its tangent lines. The tangent line goes through a point \((-d, 0)\). Find the equation of the tangent line.

\[ \text{Figure 5.10. The graph of a function and its tangent line for Example 5.15.} \]

Solution: Finding the point of tangency \(x_0\) is part of the problem in this case too. We use the same approach, and employ facts (1) and (2) from Example 5.14. We also use, for the specific function in this example,
\[ f(x_0) = C \frac{x_0}{x_0 + a} \Rightarrow f'(x_0) = C \frac{a}{(x_0 + a)^2}. \]
(See Problem 11 in Chapter 3). Hence, the equation of the tangent line is
\[ y = f(x_0) + f'(x_0)(x - x_0) = C \frac{x_0}{x_0 + a} + C \frac{a}{(x_0 + a)^2}(x - x_0). \]
We can simplify this equation by factoring to obtain:
\[ y = \frac{C}{x_0 + a} \left( x_0(x_0 + a) + a(x - x_0) \right) = \frac{C}{x_0 + a} \left( x_0^2 + ax \right). \]
It is important to realize that in this equation, \(x_0, C\) and \(a\) represent fixed (known) constants, and only \(x, y\) are variables. This means that the equation expresses a linear relationship between \(x\) and \(y\), as appropriate for a straight line.

We know that the point \((-d, 0)\) is on this line, so that (plugging in \(x = -d, y = 0\)), we obtain
\[ 0 = \frac{C}{x_0 + a} \left( x_0^2 - ad \right). \]
Solving for \(x_0\) leads to \(x_0 = \sqrt{ad}\). Moreover, we can now find the equation of the tangent line in terms of these parameters.
\[ y = \frac{C}{\sqrt{ad} + a} (ad + ax). \]
This can be simplified by factoring \( a \) from numerator and denominator to obtain

\[
y = \frac{C}{\sqrt{(d/a) + 1}} (d + x).
\]

We can easily see that when \( x = -d \), we get \( y = 0 \), as required. This forms one check that our calculations are correct.
5.1. Find the equation of the tangent line to the function \( y = f(x) = |x + 1| \) at:

(a) \( x = -1 \),
(b) \( x = -2 \),
(c) \( x = 0 \).

If there is a problem finding a tangent line at one of these points, indicate what the problem is.

5.2. A function \( f(x) \) satisfies \( f(1) = -1 \) and \( f'(1) = 2 \). What is the equation of the tangent line of \( f(x) \) at \( x = 1 \)?

5.3. Shown in Figure 5.11 is the graph of \( y = x^2 \) with one of its tangent lines.

(a) Show that the slope of the tangent to the curve \( y = x^2 \) at the point \( x = a \) is \( 2a \).

(b) Suppose that the tangent line intersects the \( x \) axis at the point (1,0). Find the coordinate, \( a \), of the point of tangency.

5.4. Shown in Figure 5.12 is the function \( f(x) = 1/x^4 \) together with its tangent line at \( x = 1 \).

(a) Find the equation of the tangent line.

(b) Determine the points of intersection of the tangent line with the \( x \) and the \( y \) axes.

(c) Use the tangent line to obtain a linear approximation to the value of \( f(1.1) \). Is this approximation larger or smaller than the actual value of the function at \( x = 1.1 \)?

5.5. **Tangent line, continued:** Shown in Figure 5.13 is the function \( f(x) = x^3 \) with a tangent line at the point (1,1).

(a) Find the equation of the tangent line.

(b) Determine the point at which the tangent line intersects the \( x \) axis.
97

(c) Compute the value of the function at \( x = 1.1 \). Compare this with the value of \( y \) on the tangent line at \( x = 1.1 \). (This latter value is the linear approximation of the function at the desired point based on its known value and known derivative at the nearby point \( x = 1 \).)

5.6. Shown in Figure 5.14 is the graph of a function and its tangent line at the point \( x_0 \).

(a) Find the equation of the tangent line expressed in terms of \( x_0 \), \( f(x_0) \) and \( f'(x_0) \).

(b) Find the coordinate \( x_1 \) at which the tangent line intersects the \( x \) axis.

5.7. **Estimating a square root:** Use Newton’s method to find an approximate value for \( \sqrt{8} \). (Hint: First think of a function, \( f(x) \), such that \( f(x) = 0 \) has the solution \( x = \sqrt{8} \).)

5.8. **Finding points of intersection:** Find the point(s) of intersection of: \( y_1 = 8x^3 - 10x^2 + x + 2 \) and \( y_2 = x^3 + 15x^2 - x - 4 \) (Hint: an intersection point exists between \( x = 3 \) and \( x = 4 \)).

5.9. **Roots of cubic equations:** Find the roots for each of the following cubic equations using Newton’s method:

(a) \( x^3 + 3x - 1 = 0 \)

(b) \( x^3 + x^2 + x - 2 = 0 \)
Figure 5.14. Figure for problem 6

(c) \( x^3 + 5x^2 - 2 = 0 \) (Hint: Find an approximation to a first root \( a \) using Newton’s method, then divide the left hand side of the equation by \( (x - a) \) to obtain a quadratic equation, which can be solved by the quadratic formula.)

5.10. The parabola \( y = x^2 \) has two tangent lines that intersect at the point \((2, 3)\). These are shown as the dark lines in Figure 5.15. [Remark: note that the point \((2, 3)\) is not on the parabola]. Find the coordinates of the two points at which the lines are tangent to the parabola.

Figure 5.15. Figure for Problem 10

5.11. **An approximation for the square root**: Use a linear approximation to find a rough estimate of the following functions at the indicated points.

(a) \( y = \sqrt{x} \) at \( x = 10 \). (Use the fact that \( \sqrt{9} = 3 \).)

(b) \( y = 5x - 2 \) at \( x = 1 \).

5.12. Use the method of linear approximation to find the cube root of

(a) \( 0.065 \) (Hint: \( \sqrt[3]{0.064} = 0.4 \))

(b) \( 215 \) (Hint: \( \sqrt[3]{216} = 6 \))

5.13. Use the data in the graph in Figure 5.16 to make the best approximation you can to \( f(2.01) \).
5.14. Approximate the value of \( f(x) = x^3 - 2x^2 + 3x - 5 \) at \( x = 1.001 \) using the method of linear approximation.

5.15. Approximate the volume of a cube whose length of each side is 10.1 cm.

5.16. Using Newton’s method to find a critical point: Consider the function

\[ g(x) = x^5 - 4x^4 + 3x^3 + x^2 - 3x. \]

Critical points of a function are defined as values of \( x \) for which \( g'(x) = 0 \). However, for this fifth order polynomial, it is not easy to find such points analytically (i.e., using pencil and paper).

(a) Use Newton’s Method to find a critical point for positive values of \( x \). Find an initial approximation for the critical point by plotting the function, but use a spreadsheet and explain how you set up the calculations. Provide an answer accurate to 8 decimal points.

(b) Explain why a starting value of \( x_0 = 1 \) for Newton’s Method does not lead to the positive critical point. You may support your argument with a graph or verbal explanation.