Chapter 15

Cycles, periods, and rates of change

15.1 Derivatives of trigonometric functions

Having acquainted ourselves with properties of the trigonometric functions and their inverses in Chapter 14, we are ready to compute their derivatives and apply our results to understanding rates of change of these periodic functions. We compute derivatives in this section, and use these results in a medley of problems on optima, related rates, and differential equations afterwards.

Section 15.1 Learning goals

1. Be able to use the definition of the derivative to calculate the derivatives of \( \sin(x) \) and \( \cos(x) \).

2. Using the quotient rule, be able to compute derivatives of \( \tan(x), \sec(x), \csc(x), \) and \( \cot(x) \).

3. Using properties of the inverse trigonometric functions and implicit differentiation, be able to calculate derivatives of \( \arcsin(x), \arccos(x), \) and \( \arctan(x) \).

15.1.1 Limits of trigonometric functions

In Chapter 3, we zoomed in on the graph of the sine function close to the origin (Fig. 3.2). By doing so, we reasoned that

\[
\sin(x) \approx x, \quad \text{for small } x.
\]

Restated, with \( h \) replacing the variable \( x \)

\[
\sin(h) \approx h, \quad \text{for small } h \quad \Rightarrow \quad \frac{\sin(h)}{h} \approx 1 \text{ for small } h
\]
Chapter 15. Cycles, periods, and rates of change

The smaller is $h$, the better this “tangent line” approximation becomes. In more formal limit notation, we say that

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1. \quad (15.1)$$

(See (3.1).) This is a very important limit, and we will apply it directly in computing the derivative of the trigonometric functions using the definition of the derivative.

A similar analysis of the graph of the cosine function, (here omitted) leads to a second important limit:

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0. \quad (15.2)$$

We can now apply these to computing derivatives of both the sine and the cosine functions.

### 15.1.2 Derivatives of sine, cosine, and other trigonometric functions

Let $y = f(x) = \sin(x)$ be the function to differentiate, where $x$ is now the independent variable (previously called $t$). Below, we use the definition of the derivative to compute the derivative of this function.

**Example 15.1 (Derivative of $\sin(x)$):** Compute the derivative of $y = \sin(x)$ using the definition of the derivative.

**Solution:** We apply the definition of the derivative as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\frac{d \sin(x)}{dx} = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x) \cos(h) + \sin(h) \cos(x) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left( \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} \right)$$

$$= \sin(x) \left( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right) + \cos(x) \left( \lim_{h \to 0} \frac{\sin(h)}{h} \right)$$

$$= \cos(x).$$

Observe that a trigonometric identity (for the sum of angles - see Eqn. (F.3)) and the two limits (15.1) and (15.2) were used to obtain the final result.

A similar calculation using the function $\cos(x)$ leads to the result

$$\frac{d \cos(x)}{dx} = - \sin(x).$$

(The same two limits appear in this calculation, as well as the trigonometric identity (F.4).) We can now calculate the derivative of the any of the other trigonometric functions using the quotient rule.
15.1. Derivatives of trigonometric functions

<table>
<thead>
<tr>
<th>$y = f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(x)$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$-\sin(x)$</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$\sec^2(x)$</td>
</tr>
<tr>
<td>$\csc(x)$</td>
<td>$-\csc(x)\cot(x)$</td>
</tr>
<tr>
<td>$\sec(x)$</td>
<td>$\sec(x)\tan(x)$</td>
</tr>
<tr>
<td>$\cot(x)$</td>
<td>$-\csc^2(x)$</td>
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</tbody>
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Table 15.1. Derivatives of the trigonometric functions

Example 15.2 (Derivative of the function $\tan(x)$): Compute the derivative of $y = \tan(x)$ using the quotient rule.

Solution: We apply the quotient rule:

$$\frac{d\tan(x)}{dx} = \frac{[\sin(x)]'\cos(x) - [\cos(x)]'\sin(x)}{\cos^2(x)}.$$ 

Using the recently found derivatives for the sine and cosine, we have

$$\frac{d\tan(x)}{dx} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}.$$ 

But the numerator of the above can be simplified using the trigonometric identity (14.1), leading to

$$\frac{d\tan(x)}{dx} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

The derivatives of the six trigonometric functions are given in Table 15.1. The reader should practice the use of the quotient rule by verifying one or more of the derivatives of the relatives $\csc(x)$ or $\sec(x)$. In practice, the most important functions are the first three, and their derivatives should be remembered, as they are frequently encountered in practical applications.

15.1.3 Derivatives of the inverse trigonometric functions

Implicit differentiation can be used to determine all derivatives of the inverse trigonometric functions. As an example, we demonstrate how to compute the derivative of $\arctan(x)$. To do so, we will need to recall that the derivative of the function $\tan(x)$ is $\sec^2(x)$. We will also use the identity $\tan^2(x) + 1 = \sec^2(x)$. (See (F.1).)

Let $y = \arctan(x)$. Then on the appropriate interval, we can replace this relationship with the equivalent one:

$$\tan(y) = x.$$ 

Differentiating implicitly with respect to $x$ on both sides, we obtain

$$\sec^2(y) \frac{dy}{dx} = 1,$$
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\[
y = f(x) \quad f'(x)
\]

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \arcsin(x) )</td>
<td>( \frac{1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>( \arccos(x) )</td>
<td>( -\frac{1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>( \arctan(x) )</td>
<td>( \frac{1}{x^2+1} )</td>
</tr>
</tbody>
</table>

Table 15.2. Derivatives of the inverse trigonometric functions.

\[
\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\tan^2(y) + 1}.
\]

Now using again the relationship \( \tan(y) = x \), we obtain

\[
\frac{d\arctan(x)}{dx} = \frac{1}{x^2 + 1}.
\]

This expression will be used frequently in integral calculus. The derivatives of the important inverse trigonometric functions are given in Table 15.2.

15.2 Changing angles and related rates

The examples in this section will allow us to practice chain rule applications using the trigonometric functions. We will discuss a number of problems, and show how the basic properties of these functions, together with some geometry are used to arrive at desired results.

Section 15.2 Learning goals

1. Understand how the chain rule is applied to problems in which geometric quantities depend on angles that are changing in time (“related rates”).

2. Given a description of the geometry and rate of change of angle or side (e.g., in a triangle) be able to set up the mathematical solution to the word-problem using the ideas of related rates, analysis of the geometry and properties of trigonometric functions (e.g., trigonometric identities).

Example 15.3 (A point on a circle:) A point moves around the rim of a circle of radius 1 so that the angle \( \theta \) subtended by the radius vector to that point changes at a constant rate,

\[
\theta = \omega t,
\]

where \( t \) is time. Determine the rate of change of the \( x \) and \( y \) coordinates of that point.
15.2. Changing angles and related rates

Solution: We have $\theta(t), x(t), \text{ and } y(t)$ all functions of $t$. (The geometry is captured by Fig. 14.3, but the angle has been renamed $\theta$, and we consider it to be time-dependent.) The fact that $\theta$ is proportional to $t$ means that

$$\frac{d\theta}{dt} = \omega.$$  

The $x$ and $y$ coordinates of the point are related to the angle by

$$x(t) = \cos(\theta(t)) = \cos(\omega t),$$

$$y(t) = \sin(\theta(t)) = \sin(\omega t).$$

This implies (by the chain rule) that

$$\frac{dx}{dt} = \frac{d\cos(\theta)}{d\theta} \frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \frac{d\sin(\theta)}{d\theta} \frac{d\theta}{dt}.$$  

Performing the required calculations, we have

$$\frac{dx}{dt} = -\sin(\theta)\omega,$$

$$\frac{dy}{dt} = \cos(\theta)\omega.$$  

We will see some interesting consequences of this in a later section.

Example 15.4 (Runners on a circular track:) Two runners start at the same position on a circular race track of length 400 meters. Joe Runner takes 50 sec, while Michael Johnson takes 43.18 sec to complete the 400 meter race. Determine the rate of change of the angle formed between the two runners and the center of the track, assuming that the runners are running at a constant rate.

Solution: We are told that the track is 400 meters in length (total). However, we will see that this information does not enter into the solution. Joe completes one cycle around the track ($2\pi$ radians) in 50 sec, while Michael completes a cycle in 43.18 sec. (This means that Joe has a period of $T_1 = 50$ sec, and a frequency of $\omega_1 = 2\pi/T_1 = 2\pi/50$ radians per sec. Similarly, Michael’s period is $T_2 = 43.18$ sec and his frequency is $\omega_2 = 2\pi/T_2 = 2\pi/43.18$ radians per sec.

Let $\theta_J, \theta_M$ be the angles subtended between one of the runners and the starting line. (We will take the $x$ axis as that starting line, by convention, as we did in Fig. 14.3.) From this, we find that

$$\frac{d\theta_J}{dt} = \frac{2\pi}{50} = 0.125 \text{ radians per sec},$$

$$\frac{d\theta_M}{dt} = \frac{2\pi}{43.18} = 0.145 \text{ radians per sec.}$$
Thus, the angle between the runners, $\theta_M - \theta_J$ changes at the rate

$$\frac{d(\theta_M - \theta_J)}{dt} = \frac{d\theta_M}{dt} - \frac{d\theta_J}{dt} = 0.145 - 0.125 = 0.02 \text{ radians per sec}.$$ 

**Example 15.5 (Simple law of cosines):** The law of cosines applies to an arbitrary triangle, as reviewed in Appendix F. (See Eqn (F.2).) Consider the triangle shown in Figure 15.1. Suppose that the angle $\theta$ increases at a constant rate, i.e. $d\theta/dt = k$. If the sides

![Diagram of triangle](image)

Figure 15.1. Law of cosines states that $c^2 = a^2 + b^2 - 2ab \cos(\theta)$.

$a = 3, b = 4$, are of constant length, determine the rate of change of the length $c$ opposite this angle at the instant that $c = 5$. 

**Solution:** Let $a, b, c$ be the lengths of the three sides, with $c$ the length of the side opposite angle $\theta$. The law of cosines states that

$$c^2 = a^2 + b^2 - 2ab \cos(\theta).$$

We identify the changing quantities by writing this relation in the form

$$c^2(t) = a^2 + b^2 - 2ab \cos(\theta(t))$$

so it is evident that only $c$ and $\theta$ will vary with time, while $a, b$ remain constant. We are also told that

$$\frac{d\theta}{dt} = k.$$ 

Differentiating and using the chain rule leads to:

$$2c \frac{dc}{dt} = -2ab \frac{d}{d\theta} \frac{d\cos(\theta)}{d\theta} \frac{d\theta}{dt}.$$ 

But $d\cos(\theta)/d\theta = -\sin(\theta)$ and $d\theta/dt = k$, so that

$$\frac{dc}{dt} = \frac{-ab}{c} (-\sin(\theta)) \frac{d\theta}{dt} = \frac{ab}{c} k \sin(\theta).$$

We now note that at the instant in question, $a = 3, b = 4, c = 5$, forming a Pythagorean triangle in which the angle opposite $c$ is $\theta = \pi/2$. We can see this fact using the law of cosines, and noting that

$$c^2 = a^2 + b^2 - 2ab \cos(\theta), \quad 25 = 9 + 16 - 24 \cos(\theta).$$
This implies that $0 = -24 \cos(\theta)$, $\cos(\theta) = 0$ so that $\theta = \pi/2$. Substituting these into our result for the rate of change of the length $c$ leads to

$$\frac{dc}{dt} = \frac{ab}{c} k = \frac{3 \cdot 4}{5} k.$$ 

**Example 15.6 (Clocks):** Find the rate of change of the angle between the minute hand and hour hand on a clock.

![Figure 15.2. Figure for Examples 15.6 and 15.7.](image)

**Solution:** We will call $\theta_1$ the angle that the minute hand subtends with the $x$ axis (horizontal direction) and $\theta_2$ the angle that the hour hand makes with the same axis.

If our clock is working properly, each hand will move around at a constant rate. The hour hand will trace out one complete revolution ($2\pi$ radians) every 12 hours, while the minute hand will complete a revolution every hour. Both hands move in a clockwise direction, which (by convention) is towards negative angles. This means that

$$\frac{d\theta_1}{dt} = -2\pi \text{ radians per hour},$$

$$\frac{d\theta_2}{dt} = \frac{2\pi}{12} \text{ radians per hour}.$$ 

The angle between the two hands is the difference of the two angles, i.e.

$$\theta = \theta_1 - \theta_2.$$ 

Thus,

$$\frac{d\theta}{dt} = \frac{d}{dt}(\theta_1 - \theta_2) = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt} = -2\pi + \frac{2\pi}{12}.$$ 

We find that the rate of change of the angle between the hands is

$$\frac{d\theta}{dt} = -2\pi \frac{11}{12} = -\pi \frac{11}{6}.$$
Example 15.7 (Clocks, continued:) Suppose that the length of the minute hand is 4 cm and the length of the hour hand is 3 cm. At what rate is the distance between the hands changing when it is 3:00 o’clock?

Solution: We use the law of cosines to give us the rate of change of the desired distance. We have the triangle shown in Figure 15.2 in which side lengths are \( a = 3 \), \( b = 4 \), and \( c(t) \) opposite the angle \( \theta(t) \). From the previous example, we have

\[
\frac{dc}{dt} = \frac{ab}{c} \sin(\theta) \frac{d\theta}{dt}.
\]

At precisely 3:00 o’clock, the angle in question is \( \theta = \frac{\pi}{2} \) and it can also be seen that the Pythagorean triangle \( abc \) leads to

\[
c^2 = a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25
\]

so that \( c = 5 \). We found from our previous analysis that \( d\theta/dt = \frac{11}{6} \pi \). Using this information leads to:

\[
\frac{dc}{dt} = \frac{3 \cdot 4}{5} \sin \left( \frac{\pi}{2} \right) \left( -\frac{11}{6} \pi \right) = -\frac{22}{5} \pi \text{ cm per hr.}
\]

The negative sign indicates that at this time, the distance between the two hands is decreasing.

15.3 The Zebra danio’s escape responses

We consider an example involving trigonometry and related rates with a biological application. We first consider the geometry on its own, and then link it to the biology of predator avoidance and escape responses.

<table>
<thead>
<tr>
<th>Section 15.3 Learning goals</th>
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</thead>
<tbody>
<tr>
<td>1. Understand the geometry of a visual angle, and determine how that angle changes as the distance to the viewed object (or the size of the object) changes (an application of “related rates”).</td>
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<tr>
<td>2. Be able to determine how the rate of change of the visual angle of a prey fish (zebra danio) changes as a predator of a given size approaches it at some speed.</td>
</tr>
<tr>
<td>3. Understand the link between the rate of change of the visual angle and the triggering of an escape response.</td>
</tr>
<tr>
<td>4. Using the results of the analysis, be able to explain in words under what circumstances the prey does (or does not) manage to escape from its predator.</td>
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</tbody>
</table>
15.3.1 Visual angles

Example 15.8 (Visual angle:) In the triangle shown in Figure 15.3, an object of height \( s \) is moving towards an observer. Its distance from the observer at some instant is labeled \( x(t) \) and it approaches at some constant speed, \( v \). Determine the rate of change of the angle \( \theta(t) \) and how it depends on speed, size, and distance of the object. Often \( \theta \) is called a visual angle, since it represents the angle that an image subtends on the retina of the observer. A more detailed example of this type is discussed in Section 15.3.2.

\[
\begin{align*}
\theta(t) & \rightarrow s \\
& \downarrow \\
x(t) &
\end{align*}
\]

Figure 15.3. A visual angle \( \theta \) would change as the distance \( x \) decreases. The size \( s \) is assumed constant. See Example 15.8.

Solution: We are given the information that the object approaches at some constant speed, \( v \). This means that

\[
\frac{dx}{dt} = -v.
\]

(The minus sign means that the distance \( x \) is decreasing.) Using the trigonometric relations, we see that

\[
\tan(\theta) = \frac{s}{x}.
\]

If the size, \( s \), of the object is constant, then the changes with time imply that

\[
\tan(\theta(t)) = \frac{s}{x(t)}.
\]

We differentiate both sides of this equation with respect to \( t \), and obtain

\[
\frac{d}{dt} \frac{\tan(\theta)}{d\theta} \frac{d\theta}{dt} = \frac{d}{dt} \left( \frac{s}{x(t)} \right),
\]

so that

\[
\sec^2(\theta) \frac{d\theta}{dt} = -\frac{1}{x^2} \frac{dx}{dt},
\]

so that

\[
\frac{d\theta}{dt} = -\frac{1}{s \sec^2(\theta)} \frac{1}{x^2} \frac{dx}{dt}.
\]

We can use the trigonometric identity

\[
\sec^2(\theta) = 1 + \tan^2(\theta)
\]
to express our answer in terms of the size, \( s \), the distance of the object, \( x \) and the speed \( v \):

\[
\sec^2(\theta) = 1 + \left(\frac{s}{x}\right)^2 = \frac{x^2 + s^2}{x^2}
\]

so

\[
\frac{d\theta}{dt} = -s \frac{x^2}{x^2 + s^2} \frac{1}{x^2} \frac{dx}{dt} = \frac{s}{x^2 + s^2} v.
\]

(Two minus signs cancelled above.) Thus, the rate of change of the visual angle is \( sv/(x^2 + s^2) \). The angle thus changes at a rate proportional to the speed of the object. However, the dependence on the size of the object is more involved, and we will explore the implications next.

### 15.3.2 The Zebra danio and a looming predator

Visual angles are important to predator avoidance. We use the ideas of Example 15.8 to consider a problem in biology, studied by Larry Dill, a biologist at Simon Fraser University in Burnaby, BC.

![Figure 15.4. A cartoon showing the visual angle, \( \alpha(t) \) and how it changes as a predator approaches its prey, the zebra danio.](image)

![Figure 15.5. The geometry of the escape response problem.](image)

The Zebra danio is a small tropical fish, which has many predators (larger fish) eager to have it for dinner. Surviving through the day means being able to sense danger quickly
enough to escape from a hungry pair of jaws. However, the danio cannot spend all its time escaping. It too, must find food, mates, and carry on activities that sustain it. Thus, a finely tuned mechanism which allows it to react to danger but avoid over-reacting would be advantageous. We investigate the visual basis of an escape response, based on a hypothesis formulated by Dill in his papers [5, 6].

Figure 15.5 shows the relation between the angle subtended at the Danio’s eye and the profile size $S$ of an approaching predator, currently located at distance $x$ away. We will assume that the predator is approaching the prey at constant speed, $v$. This means that the distance $x$ is decreasing, so that its rate of change satisfies

$$\frac{dx}{dt} = -v.$$ 

next, we use this information and geometry to characterize the rate of change of the angle $\alpha$.

**Example 15.9 (Danio’s visual angle)** Use the geometry and information given above to express the rate of change of the angle $\alpha$ in terms of the size and speed of the approaching predator, and its distance away from its prey, the danio.

**Solution:** If we consider the top half of the triangle shown in Figure 15.5 we find a Pythagorean triangle identical to the one we have seen in Example 15.8 provided we re-define $\theta = \alpha/2$, $s = S/2$. The side labeled $x$ is identical in both pictures. Thus, the trigonometric relation that holds is:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{(S/2)}{x}. \quad (15.3)$$

Furthermore, based on the results of Example 15.8, we know that $d\alpha/dt$ can be written as

$$\frac{d\alpha}{dt} = \frac{S}{x^2 + (S/2)^2}v = \frac{Sv}{x^2 + (S^2/4)}. \quad (15.4)$$

**Example 15.10 (Distance-dependence)** Use the relationship in Eqn. (15.4) to sketch a rough graph of the rate of change of the visual angle $\alpha$ versus the distance $x$ of the predator.

**Solution:** We are asked to sketch $d\alpha/dt$ versus $x$. Let us denote by $f(x)$ the function of $x$ that we want to graph. Then from (15.4),

$$f(x) \equiv \frac{Sv}{x^2 + (S^2/4)}.$$ 

We first make three observations about this function.

- When $x = 0$, i.e., when the predator has reached its prey, we have that

$$f(x) = \frac{Sv}{0 + (S^2/4)} = \frac{4v}{S}.$$

This provides the “y intercept” of the graph.
Since the variable $x$ appears only in the denominator, the function $f(x)$ is always decreasing.

For $x \to \infty$, when the predator is very far away, we have a large value $x^2$ in the denominator, so

$$f(x) \to 0.$$ 

These observations help us to construct an informative graph. A sketch of the rate of change of visual angle, $d\alpha/dt$, versus the current distance $x$ of the predator away from the prey is shown in Figure 15.7.

![Figure 15.6](image)

**Figure 15.6.** The function $f(x)$ plotted against $x$. This graph shows that the rate of change of visual angle $d\alpha/dt \equiv f(x)$ is small when the distance to the predator $x$ is large.

**When to escape?**

What sort of visual input should the danio respond to, if it is to be efficient at avoiding the predator? In principle, we would like to consider a response that has the following features

- If the predator is too far away, if it is moving slowly, or if it is moving in the opposite direction, it should appear harmless and should not cause undue panic and inappropriate escape response, since this uses up the prey’s energy to no good purpose.

- If the predator is coming quickly towards the danio, and approaching directly, it should be perceived as a threat and should trigger the escape response.

In keeping with these reasonable expectations, the hypothesis proposed by Dill is that: 

**The escape response is triggered when the predator approaches so quickly, that the rate of change of the visual angle is greater than some critical value.**

We will call that critical value $K_{\text{crit}}$. This constant would depend on how “skittish” the Danio is, given factors such as perceived risks of its environment. This means that the escape response is triggered in the Danio when

$$\frac{d\alpha}{dt} = K_{\text{crit}}.$$ 

**Example 15.11 (Finding the predator’s distance)** How far away will the predator be when this alarm response is triggered?
15.3. The Zebra danio’s escape responses

Solution: We rewrite the above condition using what we have determined for the dependence of \( d\alpha /dt \) on the geometric quantities in the problem. We obtain

\[ K_{\text{crit}} = \frac{Sv}{x^2 + (S^2/4)}. \]  

(15.5)

We now ask what value of the distance \( x \) satisfies this equation. Figure 15.7(a) illustrates a geometric solution to this equation. We show the line \( y = K_{\text{crit}} \) and the curve \( y = Sv/(x^2 + (S^2/4)) \) superimposed on the same coordinate system. The value of \( x \), labeled \( x_{\text{react}} \) will be the distance of the predator at the instant that the Danio realizes that it is under threat and should escape. We can determine the value of this distance, referred to as the reaction distance, by solving for \( x \), obtaining (see Exercise 11 in this chapter):

\[ x_{\text{react}} = \sqrt{\frac{Sv}{K_{\text{crit}}} - \frac{S^2}{4}} = \sqrt{S \left( \frac{v}{K_{\text{crit}}} - \frac{S}{4} \right)}. \]  

(15.6)

Example 15.12 (Lunch) Interpret the reaction distance \( x_{\text{react}} \). Are there ever cases in which the prey does not notice a predator in time to escape?

Solution: Figure 15.7(b) illustrates a possibility where there is no distance at which which \( K_{\text{crit}} = Sv/(x^2 + (S^2/4)) \). (An intersection point satisfying this equality fails to exist.) This may happen if either the Danio has a very high threshold of alert, so that it fails to react to threats, or if the curve depicting \( d\alpha /dt \) is too low. That happens either if \( S \) is very large (big predator) or if \( v \) is small (slow moving predator “sneaking up” on its prey). From this scenario, we find that in some situations, the fate of the Danio would be sealed in the jaws of its pursuer.

Large slow predators beat Danio’s escape response

It is clear that the reaction distance of the Danio with reaction threshold \( K_{\text{crit}} \) would be greatest for certain sizes of predators. In Figure 15.8, we plot the reaction distance \( x_{\text{react}} \) versus the predator size \( S \). We see that for very small predators \( (S \approx 0) \) or large predators \( (S \approx 4v/K_{\text{crit}}) \) the distance at which escape response is triggered is very small. This means...
that the Danio may miss noticing such predators until they are too close for a comfortable escape, resulting in calamity. Some predators will be detected when they are very far away (large $x_{\text{react}}$).

**Example 15.13 (Bad design for a predator)** Some predators are more easily detected than others. Find the size of predator for which the reaction distance is maximal, and interpret your finding.

**Solution:** We solve this problem using differentiation (Exercise 12 in this chapter) and find that $x_{\text{react}}$ has a critical point at $S = 2v/K_{\text{crit}}$. From Fig. 15.8, we see that this critical point is a local maximum. We can also reason from the expression in Eqn. (15.6): clearly $x_{\text{react}}$ cannot be negative. However, we see that $x_{\text{react}} = 0$ at $S = 0$ and at $S = 4v/K_{\text{crit}}$. Hence, $x_{\text{react}}$ has a local maximum for some predator size between these two values. In short, a predator of size $S = 2v/K_{\text{crit}}$ would be detected as far away as possible (largest possible $x_{\text{react}}$), giving the prey a good chance to escape.

We also observe that at sizes $S > 4v/K_{\text{crit}}$, the reaction distance is not defined at all. We have seen this from Figure 15.7(b): when $K_{\text{crit}} > 4v/S$, the straight line and the curve fail to intersect, and there is no solution.

Figure 15.8(b) illustrates the dependence of the reaction distance $x_{\text{react}}$ on the speed $v$ of the predator. We find that for small values of $v$, $x_{\text{react}}$ is not defined: the Danio would not notice the threat posed by predators that swim very slowly. (See Exercise 13 for the largest velocity that fails to trigger the escape response.)
15.3.3 Alternate approach involving inverse trig functions

In Section 15.3, we studied the escape response of the zebra danio and showed that the connection between the visual angle and distance to predator satisfies

\[ \tan \left( \frac{\alpha}{2} \right) = \frac{(S/2)}{x}. \]

We also computed the rate of change of the visual angle per unit time using implicit differentiation and related rates. But there are various approaches to solve a mathematical problem. Here we illustrate that an alternate approach is to express the relationship of interest in terms of inverse trigonometric functions, and then use the derivative of that function to find the desired rate of change\(^{47}\). This will provide practice with differentiation of inverse trigonometric functions.

**Example 15.14** Use the inverse function \( \arctan \) to restate the angle \( \alpha \) in Eqn. 15.7 as a function of \( x \). Then differentiate that function using the chain rule to compute \( d\alpha/dt \).

**Solution:** We can restate this relationship using the inverse trigonometric function \( \arctan \) as follows:

\[ \frac{\alpha}{2} = \arctan \left( \frac{S}{2x} \right). \]

Our experience with the derivative of this function will be useful below. Since both the angle \( \alpha \) and the distance from the predator \( x \) change with time, we indicate so by writing

\[ \alpha(t) = 2 \arctan \left( \frac{S}{2x(t)} \right). \]

We apply the chain rule to this expression to calculate the rate of change of the angle \( \alpha \) with respect to time. Let \( u = S/2x \). Recall that \( S \) is a constant. Then the derivative of the inverse trigonometric function,

\[ \frac{d \arctan(u)}{du} = \frac{1}{u^2 + 1}, \]

and the chain rule leads to

\[ \frac{d\alpha(t)}{dt} = \frac{d \arctan(u)}{du} \frac{du}{dx} \frac{dx}{dt} = \frac{1}{u^2 + 1} \left( -\frac{S}{2x^2(t)} \right) (-v). \]

By simplifying, we arrive at the same result, namely that

\[ \frac{d\alpha}{dt} = \frac{Sv}{x^2 + (S^2/4)}. \]

This is the rate of change of the visual angle, and agrees with Example 15.8.

\(^{47}\)This section is optional and can be skipped or left as an independent exercise for the student.
Chapter 15. Cycles, periods, and rates of change

15.4 For further study: Trigonometric functions and differential equations

As we have seen in this chapter, the functions $\sin(t)$ and $\cos(t)$ are related to one another via differentiation: one is the derivative of the other (with a multiple of the factor $(-1)$):

$$\frac{d}{dt}\sin(t) = \cos(t), \quad \frac{d}{dt}\cos(t) = -\sin(t).$$

The connection becomes even clearer when we examine the second derivatives of these functions:

$$\frac{d^2}{dt^2}\sin(t) = -\sin(t), \quad \frac{d^2}{dt^2}\cos(t) = -\sin(t).$$

Thus, for each of the functions $y = \sin(t), y = \cos(t)$, we find that the function and its second derivative are related to one another by the differential equation (DE) $\frac{d^2y}{dt^2} = -y$. Here the highest derivative is a second derivative, and we denote this as a second order DE.

More generally, we make the following observations. These follow by the same reasoning, where the chain rule is applied in differentiation.

<table>
<thead>
<tr>
<th>The functions $x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t)$ satisfy a pair of differential equations,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dx}{dt} = -\omega y, \quad \frac{dy}{dt} = \omega x.$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The functions $x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t)$ also satisfy a related differential equation with a second derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d^2x}{dt^2} = -\omega^2 x.$</td>
</tr>
</tbody>
</table>

Students of physics will here recognize the equation that governs the behaviour of a harmonic oscillator, and will see the connection between the circular motion of our point on the circle, and the differential equation for periodic motion.
Exercises

15.1. Calculate the first derivative for the following functions.

(a) \( y = \sin x^2 \)
(b) \( y = \sin^2 x \)
(c) \( y = \cot^2 \sqrt{x} \)
(d) \( y = \sec(x - 3x^2) \)
(e) \( y = 2x^3 \tan x \)
(f) \( y = \frac{x}{\cos x} \)
(g) \( y = x \cos x \)
(h) \( y = e^{-\sin^2 \frac{1}{x}} \)
(i) \( y = (2 \tan 3x + 3 \cos x)^2 \)
(j) \( y = \cos(\sin x) + \cos x \sin x \)

15.2. Take the derivative of the following functions.

(a) \( f(x) = \cos(\ln(x^4 + 5x^2 + 3)) \)
(b) \( f(x) = \sin(\sqrt{\cos^2(x) + x^3}) \)
(c) \( f(x) = 2x^3 + \log_3(x) \)
(d) \( f(x) = (x^2e^x + \tan(3x))^4 \)
(e) \( f(x) = x^2 \sqrt{\sin^3(x) + \cos^3(x)} \)

15.3. A point is moving on the perimeter of a circle of radius 1 at the rate of 0.1 radians per second. How fast is its \( x \) coordinate changing when \( x = 0.5 \)? How fast is its \( y \) coordinate changing at that time?

15.4. The derivatives of the two important trig functions are \( [\sin(x)]' = \cos(x) \) and \( [\cos(x)]' = -\sin(x) \). Use these derivatives to answer the following questions.
Let \( f(x) = \sin(x) + \cos(x), \ 0 \leq x \leq 2\pi \)
(a) Find all intervals where \( f(x) \) is increasing.
(b) Find all intervals where \( f(x) \) is concave up.
(c) Locate all inflection points.
(d) Graph \( f(x) \).

15.5. Find all points on the graph of \( y = \tan(2x), \ -\frac{\pi}{4} < x < \frac{\pi}{4} \), where the slope of the tangent line is 4.

15.6. A “V” shaped formation of birds forms a symmetric structure in which the distance from the leader to the last birds in the V is \( r = 10 \text{m} \), the distance between those trailing birds is \( D = 6 \text{m} \) and the angle formed by the V is \( \theta \), as shown in Figure 15.9 below. Suppose that the shape is gradually changing: the trailing birds start to get closer so that their distance apart shrinks at a constant rate \( dD/dt = -0.2 \text{m/min} \).
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while maintaining the same distance from the leader. (Assume that the structure is always in the shape of a V as the other birds adjust their positions to stay aligned in the flock.) What is the rate of change of the angle \( \theta \)?

15.7. A hot air balloon on the ground is 200 meters away from an observer. It starts rising vertically at a rate of 50 meters per minute. Find the rate of change of the angle of elevation of the observer when the balloon is 200 meters above the ground.

15.8. A ladder of length \( L \) is leaning against a wall so that its point of contact with the ground is a distance \( x \) from the wall, and its point of contact with the wall is at height \( y \). The ladder slips away from the wall at a constant rate \( C \).

(a) Find an expression for the rate of change of the height \( y \).

(b) Find an expression for the rate of change of the angle \( \theta \) formed between the ladder and the wall.

15.9. A cannon-ball fired by a cannon at ground level at angle \( \theta \) to the horizon \((0 \leq \theta \leq \pi/2)\) will travel a horizontal distance (called the range, \( R \)) given by the formula below:

\[
R = \frac{1}{16} v_0^2 \sin \theta \cos \theta.
\]

Here \( v_0 > 0 \), the initial velocity of the cannon-ball, is a fixed constant and air resistance is neglected. (See Figure 15.10.) What is the maximum possible range?

![Figure 15.9. Figure for Problem 6](image)

![Figure 15.10. Figure for problem 9](image)
15.10. A wheel of radius 1 meter rolls on a flat surface without slipping. The wheel moves from left to right, rotating clockwise at a constant rate of 2 revolutions per second. Stuck to the rim of the wheel is a piece of gum, (labeled $G$); as the wheel rolls along, the gum follows a path shown by the wide arc (called a “cycloid curve”) in Figure 15.11. The $(x, y)$ coordinates of the gum $(G)$ are related to the wheel’s angle of rotation $\theta$ by the formulae

$$x = \theta - \sin \theta,$$

$$y = 1 - \cos \theta,$$

where $0 \leq \theta \leq 2\pi$. How fast is the gum moving horizontally at the instant that it reaches its highest point? How fast is it moving vertically at that same instant?

![Figure 15.11. Figure for Problem 10](image)

15.11. **Zebra danio’s reaction distance**: Solve Eqn. (15.5) for $x$ and show that you get the reaction distance $x \equiv x_{\text{react}}$ given in (15.6).

15.12. **Bad design for a predator**: Some predators are more easily detected than others. Use Eqn. (15.6) to find the size of predator for which the reaction distance is maximal.

15.13. **Sneaking up on the prey**: Use Eqn. (15.6) to show that a predator moving “slowly enough” can sneak up on the prey without being detected. What is the largest velocity for which a predator of size $S$ will not be detected by a prey that responds to a visual sighting when the rate of change of the visual angle exceeds the threshold $K_{\text{crit}}$?

15.14. Find the first derivative of the following functions.

(a) $y = \arcsin x^\frac{1}{2}$

(b) $y = (\arcsin x)^\frac{1}{2}$

(c) $\theta = \arctan(2r + 1)$

(d) $y = x \arccsc \frac{1}{x}$

(e) $y = \frac{a}{x} \sqrt{a^2 - x^2} - \arcsin \frac{a}{x}, a > 0.$

(f) $y = \arccos \frac{2t}{1+t^2}$
15.15. In Figure 15.12, the point P is connected to the point O by a rod 3 cm long. The wheel rotates around O in the clockwise direction at a constant speed, making 5 revolutions per second. The point Q, which is connected to the point P by a rod 5 cm long, moves along the horizontal line through O. How fast and in what direction is Q moving when P lies directly above O? (Remember the law of cosines: \( c^2 = a^2 + b^2 - 2ab \cos \theta \).)

![Figure 15.12. Figure for Problem 15](image)

15.16. A ship sails away from a harbor at a constant speed \( v \). The total height of the ship including its mast is \( h \). See Figure 15.13.

(a) At what distance away will the ship disappear below the horizon?

(b) At what rate does the top of the mast appear to drop toward the horizon just before this? (Note: In ancient times this effect lead people to conjecture that the earth is round (radius \( R \)), a fact which you need to take into account in solving the problem.)

![Figure 15.13. Figure for Problem 16](image)

15.17. Find \( \frac{dy}{dx} \) using implicit differentiation.

(a) \( y = 2 \tan(2x + y) \)

(b) \( \sin y = -2 \cos x \)

(c) \( x \sin y + y \sin x = 1 \)
15.18. Use implicit differentiation to find the equation of the tangent line to the following curve at the point \((1, 1)\):

\[ x \sin(xy - y^2) = x^2 - 1 \]

15.19. The function \(y = \arcsin(ax)\) is a so-called inverse trigonometric function. It expresses the same relationship as does the equation \(ax = \sin(y)\). (However, this function is defined only for values of \(x\) between \(1/a\) and \(-1/a\).) Use implicit differentiation to find \(y'\).

15.20. Your room has a window whose height is 1.5 meters. The bottom edge of the window is 10 cm above your eye level. (See Figure 15.14.) How far away from the window should you stand to get the best view? (“Best view” means the largest visual angle, i.e. angle between the lines of sight to the bottom and to the top of the window.)

![Figure 15.14. Figure for Problem 20](image)

15.21. You are directly below English Bay during a summer fireworks event and looking straight up. A single fireworks explosion occurs directly overhead at a height of 500 meters. (See Figure 15.15.) The rate of change of the radius of the flare is 100 meters/sec. Assuming that the flare is a circular disk parallel to the ground, (with its center right overhead) what is the rate of change of the visual angle at the eye of an observer on the ground at the instant that the radius of the disk is \(r = 100\) meters? (Note: the visual angle will be the angle between the vertical direction and the line between the edge of the disk and the observer).

15.22. Match the differential equations given in parts (i-iv) with the functions in (a-f) which are solutions for them. (Note: each differential equation may have more than one solution)

**Differential equations:**

(i) \[ \frac{d^2y}{dt^2} = 4y \]

(ii) \[ \frac{d^2y}{dt^2} = -4y \]

(iii) \[ \frac{dy}{dt} = 4y \]

(iv) \[ \frac{dy}{dt} = -4y \]

**Solutions:**
(a) \( y(t) = 4 \cos(t) \)
(b) \( y(t) = 2 \cos(2t) \)
(c) \( y(t) = 4e^{-2t} \)
(d) \( y(t) = 5e^{2t} \)
(e) \( y(t) = \sin(2t) - \cos(2t) \)
(f) \( y(t) = 2e^{-4t} \).

15.23. **Periodic motion:**

(a) Show that the function \( y(t) = A \cos(wt) \) satisfies the differential equation

\[
\frac{d^2y}{dt^2} = -w^2y
\]

where \( w > 0 \) is a constant, and \( A \) is an arbitrary constant. [Remark: Note that \( w \) corresponds to the frequency and \( A \) to the amplitude of an oscillation represented by the cosine function.]

(b) It can be shown using Newton’s Laws of motion that the motion of a pendulum is governed by a differential equation of the form

\[
\frac{d^2y}{dt^2} = -\frac{g}{L} \sin(y),
\]

where \( L \) is the length of the string, \( g \) is the acceleration due to gravity (both positive constants), and \( y(t) \) is displacement of the pendulum from the vertical. What property of the sine function is used when this equation is approximated by the Linear Pendulum Equation:

\[
\frac{d^2y}{dt^2} = -\frac{g}{L} y.
\]

(c) Based on this Linear Pendulum Equation, what function would represent the oscillations? What would be the frequency of the oscillations?
Exercises

(d) What happens to the frequency of the oscillations if the length of the string is doubled?

15.24. Jack and Jill have an on-again off-again love affair. The sum of their love for one another is given by the function \( y(t) = \sin(2t) + \cos(2t) \).
(a) Find the times when their total love is at a maximum.
(b) Find the times when they dislike each other the most.

15.25. Let \( y = f(t) = e^{-t} \sin t, \quad -\infty < t < \infty \).
(a) Show that \( y \) satisfies the differential equation \( y'' + 2y' + 2y = 0 \).
(b) Find all critical points of \( f(t) \).