Chapter 14
Periodic and trigonometric functions

Nature abounds with examples of cyclic processes. Perhaps the most familiar and earliest one we encounter is the continually repeating heartbeat that accompanies us through life. Powering the heart are electrically active muscles whose rhythmic contractions pump blood to oxygenate tissues throughout the body. That electrical activity can be recorded on the surface of the body by an electrocardiogram (ECG), as shown in Fig. 14.1. In a normal healthy human, the electrical activity pattern associated with a complete cycle repeats itself with roughly 1 heartbeat per second at rest. At exercise, the beating heart pumps faster, so the pattern repeats more frequently.

![ECG waveform](image)

**Figure 14.1.** An electrocardiogram (ECG) shows the pattern of electrical activity of the heart. Vertical axis: millivolts, horizontal axis: time in seconds. At rest (top), the heart beats approximately 60 times per minute, but in exercise (lower trace), the heart rate increases.

It is natural to want to understand behaviour shown in Fig 14.1, in both health and disease. To do so, we must also develop a language to describe such periodic phenomena. For example, we need to quantify what is meant by “more frequent repetition” of a heart
beat, “skipping a beat” or other shift in the pattern of this or of any other cycling system.

Before trying to understand intricate examples such as ECG’s, we begin with simple prototypes of periodic functions: the trigonometric functions sine and cosine. We will become familiar with their properties and gain intuition. Much of the language and many ideas developed in this elementary context will then be helpful in the goal of analyzing periodic functions in general, and periodicity of the ECG’s in particular. (We return to this in Example 14.3.)

We devote most of this chapter to the trigonometric functions and their properties. This important new class of functions will be introduced here; their basic properties and interconnections will be discussed. Belonging to a wider class of periodic functions, these illustrate the ideas of amplitude, frequency, period, and phase. We will find that many cyclic phenomena can be described approximately by suitably adjusted basic functions such as sine and cosine. As a second theme, we return to the idea of inverse functions and show that important restrictions must be applied to ensure the existence of an inverse, particularly for the trigonometric functions. Then, in the next chapter, we calculate the derivatives of trigonometric functions and show applications to rates of change of periodic phenomena or changing angles.

### 14.1 Basic trigonometry

Trigonometric functions are closely associated with angles and ratios of sides of a right-angle triangle. But they are also connected to motion of a point around a unit circle. Before we can understand these connections, we agree on a universal way of measuring angles, and then define the functions of interest.

#### Section 14.1 Learning goals

1. Understand the definition of the **radian** as a measure for angles.
2. Understand the correspondence between a point moving on a unit circle and the sine and cosine of the angle it forms at the origin.
3. Be able to make correspondence between ratios of sides of a Pythagorean triangle and the trigonometric functions of one of its angles.
4. Review properties of the functions \( \sin(x) \) and \( \cos(x) \) and other trigonometric functions. Understand and be able to state and apply the connections between these functions (“trigonometric identities”).

#### 14.1.1 Angles and circles

Angles can be measured in a number of ways. One way is to assign a value in **degrees**, with the convention that one complete revolution is represented by 360°. Why 360? And what is a degree exactly? Is this some universal measure that any intelligent being (say on Mars or elsewhere) would find appealing? Actually, 360 is a rather arbitrary convention that arose
historically, and has no particular meaning. We could as easily have had mathematical ancestors that decided to divide circles into 1000 “equal pieces” or 240 or some other subdivision. It turns out that this measure is not particularly convenient, and we will replace it by a more universal quantity.

The universal quantity stems from the fact that circles of all sizes have one common geometric feature: they have the same ratio of circumference, to diameter, no matter what their size (or where in the universe they occur). We call that ratio $\pi$, that is

$$\pi = \frac{\text{Circumference of circle}}{\text{Diameter of circle}}$$

By construction, the diameter $D$ of a circle is a distance that corresponds to twice the radius $r$ of that circle, so

$$D = 2r.$$ 

This naturally leads to the familiar relationship of circumference $C$, to radius

$$C = 2\pi r.$$ 

(But we should not forget that this is merely a definition of the constant $\pi$. The more interesting conclusion that develops from this definition is that the area of the circle is $A = \pi r^2$, but we shall see the reason for this later, in the context of areas and integration.)

![Figure 14.2](image.png)

**Figure 14.2.** The angle $\theta$ in radians is related to the radius $R$ of the circle, and the length of the arc $S$ shown by the simple formula, $S = R\theta$.

From Figure 14.2 we see that there is a correspondence between the angle ($\theta$) subtended in a circle of given radius and the length of arc along the edge of the circle. For a circle of radius $R$ and angle $\theta$ we will define the arclength, $S$ by the relation

$$S = R\theta,$$

where $\theta$ is measured in a convenient way that we will now select. First, let us observe that both $S$ and $R$ are quantities that carry units of “distance” or “length” (for example, inches, centimeters, or miles). Then we also notice that

$$\theta = \frac{S}{R},$$

so the units in numerator and denominator cancel, and the angle $\theta$ is dimensionless quantity, i.e., is a number that carries no units.
Now consider a circle of radius $R = 1$ (called a unit circle) and denote by $s$ a length of arc around the perimeter of this unit circle. Then

$$\theta = \frac{s}{1}$$

In particular, for one complete revolution around the circle, the arclength is $s = 2\pi \cdot 1 = 2\pi$, which is just the circumference of the unit circle. In that case, it makes sense to consider the angle corresponding to one revolution as

$$\theta = \frac{2\pi}{1} = 2\pi.$$ 

This leads naturally to the definition of the radian: we identify an angle of $2\pi$ radians with one complete revolution around the circle. Note that (like degrees or other measures of angles), a radian is a number that carries no conventional “units” such as centimeter or second.

We can now use this measure for angles to assign values to any fraction of a revolution, and thus, to any angle. For example, an angle of $90^\circ$ corresponds to one quarter of a revolution around the perimeter of a unit circle, so we identify the angle $\pi/2$ radians with it. One degree is $1/360$ of a revolution, corresponding to $2\pi/360$ radians, and so on.

Let us summarize the properties of radians:

1. The length of an arc along the perimeter of a circle of radius $R$ subtended by an angle $\theta$ is $S = R\theta$ where $\theta$ is measured in radians.
2. Restated, an angle in radians is the ratio of the arclength it subtends in a circle to the radius of that circle (and hence, a radian carries no units).
3. One complete revolution, or one full cycle corresponds to an angle of $2\pi$ radians.

It is easy to convert between degrees and radians if we remember that $360^\circ$ corresponds to $2\pi$ radians. ($180^\circ$ then corresponds to $\pi$ radians, $90^\circ$ to $\pi/2$ radians, etc.)

### 14.1.2 Defining the trigonometric functions $\sin(x)$ and $\cos(x)$

Consider a point $(x, y)$ moving around the rim of a circle of radius 1, and let $t$ be some angle (measured in radians) formed by the $x$ axis and the radius vector to the point $(x, y)$ as shown in Figure 14.3. We define the functions sine and cosine, both dependent on the angle $t$ (abbreviated $\sin(t)$ and $\cos(t)$) as follows:

$$\sin(t) = \frac{y}{1} = y, \quad \cos(t) = \frac{x}{1} = x$$

That is, the function sine tracks the $y$ coordinate of the point as it moves around the unit circle, and the function cosine tracks its $x$ coordinate. (Remark: see also the review definitions of these trigonometric quantities as shown in Figure F.1 of Appendix F as the opposite over hypotenuse and adjacent over hypotenuse in a right angle triangle. The hypotenuse in our diagram is simply the radius of the circle, which is 1 by assumption.)
14.1. Basic trigonometry

Figure 14.3. Shown above is the circle of radius 1, whose equation is \( x^2 + y^2 = 1 \). The radius vector that ends at the point \((x, y)\) subtends an angle \( t \) (radians) with the x axis. The triangle is also shown enlarged to the right, where the lengths of all three sides are labeled. The trigonometric functions are just ratios of two sides of this triangle.

14.1.3 Properties of \( \sin(x) \) and \( \cos(x) \)

We now explore the consequences of these definitions:

Values of sine and cosine

- The radius of the unit circle is 1. This means that the \( x \) coordinate cannot be larger than 1 or smaller than -1. The same holds for the \( y \) coordinate. Thus the functions \( \sin(t) \) and \( \cos(t) \) are always swinging between -1 and 1. \((-1 \leq \sin(t) \leq 1 \) and \(-1 \leq \cos(t) \leq 1 \) for all angles \( t \)). The peak (maximum) value of each function is 1, the minimum is -1, and the average value is 0.

- We will adopt the convention that when the radius vector points along the \( x \) axis, the angle is \( t = 0 \), and coordinates of the point are \( x = 1, y = 0 \). This implies that \( \cos(0) = 1, \sin(0) = 0 \).

- When the radius vector points up the \( y \) axis, the angle is \( \pi/2 \) (corresponding to one quarter of a complete revolution), and coordinates of the point are \( x = 0, y = 1 \) so that \( \cos(\pi/2) = 0, \sin(\pi/2) = 1 \).

- Using simple geometry, we can also determine the lengths of all sides, and hence the ratios of the sides in a few particularly simple triangles, namely equilateral triangles (in which all angles are \( 60^\circ \)), and right triangles with two equal angles of \( 45^\circ \). These types of calculations (omitted here) lead to some easily determined values for the sine and cosine of such special angles. These values are shown in Table F.1 of the Appendix F.
Properties of sine and cosine

- Both $\sin(t)$ and $\cos(t)$ will go through the same values every time the angle $t$ completes another cycle around the circle. We refer to such functions as **periodic** functions.

- The two functions, sine and cosine depict the same underlying motion, viewed from two perspectives: $\cos(t)$ represents the projection of the circularly moving point onto the $x$ axis, while $\sin(t)$ is the projection of the same point onto the $y$ axis. In this sense, the functions are “twins”, and we can expect many relationships connecting them, as we will see.

- The cosine has its largest value at the beginning of the cycle, when $t = 0$ (since $\cos(0) = 1$), while sine has its peak value a little later, ($\sin(\pi/2) = 1$). Throughout their circular race, the sine function is $\pi/2$ radians ahead of the cosine, that is,

$$\cos(t) = \sin(t + \frac{\pi}{2}).$$

See Fig. 14.4 for graphs of both functions showing this shift by $\pi/2$.

- The period $T$ of the sine function $\sin(t)$ is defined as the value of $t$ for which one whole cycle (around the circle) has been completed. Accordingly, this period is $T = 2\pi$. Similarly the period of the cosine function $\cos(t)$ is also $2\pi$. (See Fig. 14.4.)

- The point $(x, y)$ is on a circle of radius 1, and, thus, its coordinates satisfy

$$x^2 + y^2 = 1.$$ 

This implies that

$$\sin^2(t) + \cos^2(t) = 1 \quad (14.1)$$

for any angle $t$. This is an important relation, (also called a **trigonometric identity**), and one that we will use quite often. See Appendix F for a review of other trigonometric identities.

- The sine and cosine functions have symmetries that we already encountered. $\sin(t)$ is an odd function (symmetric about the origin) and the $\cos(t)$ is an even function (symmetric about the $y$ axis). These symmetries also imply that $\sin(-t) = -\sin(t)$ and $\cos(-t) = \cos(t)$.

### 14.1.4 Other trigonometric functions

Although we shall mostly be concerned with the two basic functions described above, several others are historically important and are encountered frequently in integral calculus. These include the following:

$$\tan(t) = \frac{\sin(t)}{\cos(t)}, \quad \cot(t) = \frac{1}{\tan(t)},$$

$$\sec(t) = \frac{1}{\cos(t)}, \quad \csc(t) = \frac{1}{\sin(t)}.$$
14.2 Periodic Functions

We review these and the identities that they satisfy in Appendix F. We also include the Law of Cosines (F.2), and angle-sum identities in the same appendix.

**Figure 14.4. Periodicity of the sine and cosine. Note that the two curves are just shifted versions of one another.**

14.2 Periodic Functions

In the previous section, we identified the period of the functions \( \sin(t) \) and \( \cos(t) \) as the value of \( t \) at which one full cycle is completed. Here we formalize the idea of a periodic function, define its period, frequency, and other important features of its graph.

**Section 14.2 Learning goals**

1. Understand the definition of a periodic function.
2. Given a periodic function, be able to determine its **period**, **amplitude** and **phase**.
3. Given a graph or description of a periodic or rhythmic process, be able to “fit” an approximate sine or cosine function with the correct period, amplitude and phase.
Definition 14.1 (Periodic function). A function is said to be periodic if

\[ f(t) = f(t + T). \]

where \( T \) is a constant that we call the period of the function. Graphically, this means that if we shift the function by a constant “distance” along the horizontal axis, we see the same picture again.

Example 14.2 As a specific example, we have already noted that the trigonometric functions are periodic. The point \((x, y)\) in Figure 14.3 will repeat its trajectory every time a revolution around the circle is complete. This happens when the angle \( t \) completes one full cycle of \( 2\pi \) radians. Thus, as expected, the trigonometric functions are periodic, that is

\[ \sin(t) = \sin(t + 2\pi), \quad \text{and} \quad \cos(t) = \cos(t + 2\pi). \]

Similarly

\[ \tan(t) = \tan(t + 2\pi), \quad \text{and} \quad \cot(t) = \cot(t + 2\pi). \]

We say that the period is \( T = 2\pi \) radians. The same applies to \( \sec(t) \) and \( \csc(t) \), that is all six trigonometric functions are periodic.

14.2.1 Phase, amplitude, and frequency

In Appendix C we review how the appearance of any function changes when we transform variables, that is, when we replace the independent variable \( x \) by \( x + a \) or multiply \( f \) by a constant \( C \). These changes shift the graph of the function in one direction or another, scale one of the axes, and so on. Using these ideas it will be straightforward to follow the basic changes in shape of a typical trigonometric function to which similar transformations are applied.

A function of the form

\[ y = f(t) = A \sin(\omega t) \]

has both its \( t \) and \( y \) axes scaled, as shown in Fig. 14.5(c). The constant \( A \), referred to as the amplitude of the graph, scales the \( y \) axis so that the oscillation swings between a low value of \(-A\) and a high value of \( A \). The constant \( \omega \), called the frequency, scales the \( t \) axis. This results in crowding together of the peaks and valleys (if \( \omega > 1 \)) or stretching them out (if \( \omega < 1 \)). One full cycle is completed when

\[ \omega t = 2\pi, \]

and this occurs at time

\[ t = \frac{2\pi}{\omega}. \]

We have already used the symbol \( T \), to denote this special time, and defined \( T \) as the period of the function. We note the connection between frequency and period:

\[ \omega = \frac{2\pi}{T}, \quad \Rightarrow \quad T = \frac{2\pi}{\omega}. \]
14.2. Periodic Functions

The sine function $y = \sin(t)$ shown in (a) is transformed in several ways. (b) Multiplying the function by a constant ($A = 2$) stretches the graph vertically. $A$ is called the amplitude. (c) Multiplying the independent variable by a constant ($\omega = 3$) increases the frequency, i.e. the number of cycles per unit time. (d) Subtracting a constant ($a = 0.8$) from the independent variable shifts the graph horizontally to the right.

If we examine a graph of function

$$y = f(t) = A \sin(\omega(t - a))$$

we find that the graph has been shifted in the positive $t$ direction by $a$, as in Fig. 14.5(d). We note that at time $t = a$, the value of the function is

$$y = f(t) = A \sin(\omega(a - a)) = A \sin(0) = 0.$$  

This tells us that the cycle “starts” with a delay, i.e. the value of $y$ goes through zero when $t = a$. 
Another common variant of the same function can be written in the form

\[ y = f(t) = A \sin(\omega t - \phi). \]

Here \( \phi \) is called the **phase shift** of the oscillation. Comparing the above two related forms, we see that they are the same if we identify \( \phi \) with \( \omega a \). The phase shift, \( \phi \) is considered to be a quantity without units, whereas the quantity \( a \) has units of time, same as \( t \). When \( \phi = 2\pi \), (which happens when \( a = 2\pi/\omega \), the graph has been moved over to the right by one full period. (Naturally, when the graph is so moved, it looks the same as it did originally, since each cycle is the same as the one before, and same as the one after.)

Some of the scaled, shifted, sine functions described here are shown in Figure 14.5.

### 14.2.2 The periodic electrocardiogram

With the terminology of periodic function in place, we can now describe the ECG pattern for both normal resting individuals and those at exercise.

Recall that at rest, the heart beats approximately once per second. Consider ECG trace on the left in Fig. 14.6. This corresponds to a single heartbeat, and so, takes 1 second from start to finish. Suppose \( t \) represents time in seconds, and let \( y = f(t) \) represent the electrical activity (in mV) at time \( t \). (The function \( y = f(t) \) is shown in Fig. 14.6). Then, since this pattern repeats 60 times every minute (60 beats per minute, i.e. one beat every second), the function \( f \) is periodic, with period \( T = 1 \) second. Then we can write

\[ f(t) = f(t + 1) \quad \text{in seconds}. \]

However, suppose that the individual starts running. Then this relationship will no longer hold, since heartbeats become more frequent, and the length of their period, \( T \), decreases. This suggests a more natural way to mark off time. Rather than seconds, suppose we define a variable that represents time measured in terms of the length of a heartbeat cycle. We will denote that variable “the cycle time” and use the notation \( \bar{\tau} \). Then the connection between clock time \( t \) and cycle time is

\[ t = \text{time in seconds} = \text{number of cycles} \cdot \text{length of 1 cycle in sec} = \bar{\tau} \cdot T. \]

Restated,

\[ \bar{\tau} = \frac{t}{T}. \]

Note that the “number of cycles” need not be an integer - for example, \( \bar{\tau} = 2.75 \) means that we are 3/4 way into the third electrical activity cycle.

Since \( f \) is a periodic function, we can “join up” its two ends and “wrap it around a circle”, as shown in the schematic on Fig. 14.6. Then successive heartbeats are depicted by traversing the circle over and over again. This suggests identifying the beginning of a cycle with 0 and the end of a cycle with \( 2\pi \). To do so, we revise our cycle time clock as follows. Define

\[ \tau = \text{number of radians traversed since time 0}. \]

Then every heartbeat corresponds to this clock adding an increment of \( 2\pi \). The connection between this cycle clock and time \( t \) in seconds is

\[ \tau = 2\pi \bar{\tau} = 2\pi \frac{t}{T} \quad (14.2) \]
14.2. Periodic Functions

Figure 14.6. One full ECG cycle (left) has been “wrapped around a circle”.

(We check that when 1 cycle is complete, \( t = T \) and \( \tau = 2\pi \), as desired.)

We can now describe the periodicity of the ECG in terms of the cycle clock by the formula

\[
f(\tau) = f(\tau + 2\pi).
\]

As a check, we show in the next example that the relationship (14.3) reduces to the familiar period and frequency notation in terms of our original time \( t \) in seconds.

**Example 14.3 (Period and frequency of heartbeat)** Use the formula we arrived at for \( \tau \) and its connection to clock time \( t \) to transform back to time \( t \) in seconds. Express the periodicity of the function \( f \) both in terms of the period \( T \) and the frequency \( \omega \) of heartbeats.

**Solution:** Start with Eqn. (14.3),

\[
f(\tau) = f(\tau + 2\pi).
\]

Now substitute \( \tau = 2\pi t/T \) from (14.3) and simplify by making a common denominator. Then

\[
\begin{align*}
f \left( \frac{2\pi t}{T} \right) &= f \left( \frac{2\pi t + 2\pi}{T} \right) \\
&= f \left( \frac{2\pi}{T}(t + T) \right)
\end{align*}
\]

We now rewrite this in terms of the frequency \( \omega = 2\pi/T \) to arrive at

\[
f(\omega t) = f(\omega(t + T)).
\]

This relationship holds for any regular heartbeat, whether at rest or exercise where the frequency of the heartbeat, \( \omega \), is related to the period (duration of 1 beat cycle) by \( \omega = 2\pi/T \).
14.2.3 Rhythmic processes

Many natural phenomena are cyclic. It is often convenient to represent such phenomena with one or another simple periodic functions, and sine and cosine can be adapted for such purposes. Given some periodic process, we will here discuss how to determine its frequency (or period), its amplitude, and possibly its phase shift. Using these, we will create a trigonometric function (sine or cosine) that approximates the desired behaviour.

To select one or another of these functions, it helps to remember that cosine starts a cycle (at \( t = 0 \)) at its peak value, while sine starts the cycle at 0, i.e., at its average value. A function that starts at the lowest point of the cycle is \(-\cos(t)\). In most cases, the choice of function to use is somewhat arbitrary, since a phase shift can correct for the phase at which the oscillation starts.

Next, we pick a constant \( \omega \) such that the trigonometric function \( \sin(\omega t) \) (or \( \cos(\omega t) \)) has the correct period. Given a period for the oscillation, \( T \), recall that the corresponding frequency is simply \( \omega = 2\pi/T \). We then select the amplitude, and horizontal and vertical shifts to complete the mission. The examples below illustrate this process.

Example 14.4 (Daylight hours:) In Vancouver, the shortest day (8 hours of light) occurs around December 22, and the longest day (16 hours of light) is around June 21. Approximate the cyclic changes of daylight through the season using the sine function.

Solution: On Sept 21 and March 21 the lengths of day and night are equal, and then there are 12 hours of daylight. (Each of these days is called an equinox). Suppose we call identify March 21 as the beginning of a yearly day-night length cycle. Let \( t \) be time in days beginning on March 21. One full cycle takes a year, i.e. 365 days. The period of the function we want is thus

\[
T = 365
\]

and its frequency is

\[
\omega = 2\pi/365.
\]

Daylight shifts between the two extremes of 8 and 16 hours: i.e. \( 12 \pm 4 \) hours. This means that the amplitude of the cycle is 4 hours. The oscillation take place about the average value of 12 hours. We have decided to start a cycle on a day for which the number of daylight hours is the average value (12). This means that the sine would be most appropriate, so the function that best describes the number of hours of daylight at different times of the year is:

\[
D(t) = 12 + 4 \sin \left( \frac{2\pi t}{365} \right)
\]

where \( t \) is time in days and \( D \) the number of hours of light.

Example 14.5 (Hormone levels:) The level of a certain hormone in the bloodstream fluctuates between undetectable concentration at 7:00 and 100 ng/ml (nanograms per millilitre) at 19:00 hours. Approximate the cyclic variations in this hormone level with the appropriate periodic trigonometric function. Let \( t \) represent time in hours from 0:00 hrs through the day.
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\textbf{Solution:} We first note that it takes one day (24 hours) to complete a cycle. This means that the period of oscillation is 24 hours, so that the frequency is

\[ \omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}. \]

The level of hormone varies between 0 and 100 ng/ml, which can be expressed as 50 ± 50 ng/ml. (The trigonometric functions are symmetric cycles, and we are here finding both the average value about which cycles occur and the amplitude of the cycles.) We could consider the time midway between the low and high points, namely 13:00 hours as the point corresponding to the upswing at the start of a cycle of the sine function. (See Figure 14.7 for the sketch.) Thus, if we use a sine to represent the oscillation, we should shift it by 13 hrs to the left.

\[ H(t) = 50 + 50 \sin \left( \frac{\pi}{12} (t - 13) \right). \]

In the expression above, the number 13 represents a shift along the time axis, and carries units of time. We can express this same function in the form

\[ H(t) = 50 + 50 \sin \left( \frac{\pi t}{12} - \frac{13\pi}{12} \right). \]

In this version, the quantity

\[ \phi = \frac{13\pi}{12} \]

is the phase shift.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hormonal_cycles.png}
\caption{Hormonal cycles. The full cycle takes 24 hrs. (Hence the period is \( T = 24 \text{hrs} \) and the frequency is \( \omega = \frac{2\pi}{24} \text{ per hour} \). The level \( H(t) \) swings between 0 and 100 ng/ml. From the given information, we see that the average level is 50 ng/ml, and that the origin of a sine curve should be at \( t = 13 \) (i.e. 1/4 of the cycle which is 6 hrs past the time point \( t = 7 \)).}
\end{figure}
In selecting the periodic function to use for this example, we could have made other choices. For example, the same periodic can be represented by any of the functions listed below:

\[ H_1(t) = 50 - 50 \sin \left( \frac{\pi}{12}(t - 1) \right), \]
\[ H_2(t) = 50 + 50 \cos \left( \frac{\pi}{12}(t - 19) \right), \]
\[ H_3(t) = 50 - 50 \cos \left( \frac{\pi}{12}(t - 7) \right). \]

All these functions have the same values, the same amplitudes, and the same periods.

**Example 14.6 (Phases of the moon):** A cycle of waxing and waning moon takes 29.5 days approximately. Construct a periodic function to describe the changing phases, starting with a “new moon” (totally dark) and ending one cycle later.

**Solution:** The period of the cycle is \( T = 29.5 \) days, so

\[ \omega = \frac{2\pi}{T} = \frac{2\pi}{29.5}. \]

For this example, we will use the cosine function, for practice. Let \( P(t) \) be the fraction of the moon showing on day \( t \) in the cycle. Then we should construct the function so that \( 0 < P < 1 \), with \( P = 1 \) in mid cycle (see Figure 14.8). The cosine function swings between the values -1 and 1. To obtain a positive function in the desired range for \( P(t) \), we will add a constant and scale the cosine as follows:

\[ \frac{1}{2}[1 + \cos(\omega t)]. \]

This is not quite right, though because at \( t = 0 \) this function takes the value 1, rather than 0, as shown in Figure 14.8. To correct this we can either introduce a phase shift, i.e. set

\[ P(t) = \frac{1}{2}[1 + \cos(\omega t + \pi)]. \]
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Then when $t = 0$, we get $P(t) = 0.5[1 + \cos \pi] = 0.5[1 - 1] = 0.$ or we can write

$$P(t) = \frac{1}{2}[1 - \cos(\omega t + \pi)],$$

which achieves the same result.

14.3 Inverse Trigonometric functions

The introduction of trigonometric functions in this chapter provides another opportunity to illustrate the roles and properties of inverse functions. In this section, we investigate inverse trigonometric functions. As in other examples, the inverse of a given function leads to exchange of the roles of the dependent and independent variables, as well as the the roles of the domain and range. Geometrically, an inverse function is obtained by reflecting the original function about the line $y = x$. However, we must take care that the resulting graph represents a true function, i.e. satisfies all the properties required of a function.

Section 14.3 Learning goals

1. Review the concept of an inverse function, and be able to apply this idea to trigonometric functions.

2. Understand the requirement of restricting the domain (of the original function) so as to be able to define its inverse. Given any of the trigonometric functions, be able to identify a suitably restricted domain on which an inverse function can be defined.

3. Be able to simplify and/or interpret the meaning of expressions involving the trigonometric and inverse trigonometric functions.

The domains of $\sin(x)$ and $\cos(x)$ are both $-\infty < x < \infty$ while their ranges are $-1 \leq y \leq 1$. In the case of the function $\tan(x)$, the domain excludes values $\pm \pi/2$ as well as angles $2n\pi \pm \pi/2$ at which the function is undefined. The range of $\tan(x)$ is $-\infty < y < \infty$.

There is one difficulty in defining inverses for trigonometric functions: the fact that these functions repeat their values in a cyclic pattern means that a given $y$ value is obtained from many possible values of $x$. For example, the values $x = \pi/2, 5\pi/2, 7\pi/2$, etc all have identical sine values $\sin(x) = 1$. We say that these functions are not one-to-one. Geometrically, this is just saying that the graphs of the trig functions intersect a horizontal line in numerous places. When these graphs are reflected about the line $y = x$, they would intersect a vertical line in many places, and would fail to be functions: the function would have multiple $y$ values corresponding to the same value of $x$, which is not allowed. The reader may recall that a similar difficulty was encountered in an earlier chapter with the inverse function for $y = x^2$.

Footnote: The material in this section could be omitted without loss of continuity in the next chapter. If this is done, the instructor can merely skip Sections 15.1.3 and 15.3.3.
We can avoid this difficulty by restricting the domains of the trigonometric functions to a portion of their graphs that does not repeat. To do so, we select an interval over which the given trigonometric function is one-to-one, i.e. over which there is a unique correspondence between values of $x$ and values of $y$. (This just mean that we keep a portion of the graph of the function in which the $y$ values are not repeated.) We then define the corresponding inverse function, as described below.

**Arcsine is the inverse of sine**

![Figure 14.9](image1.png)

**Figure 14.9.** (a) The original trigonometric function, $\sin(x)$, in black, as well as the portion restricted to a smaller domain, $\text{Sin}(x)$, in red. The red curve is shown again in part b. (b) Relationship between the functions $\text{Sin}(x)$, defined on $-\pi/2 \leq x \leq \pi/2$ (in red) and $\arcsin(x)$ defined on $-1 \leq x \leq 1$ (in blue). Note that one is the reflection of the other about the line $y = x$. The graphs in parts (a) and (b) are not on the same scale.

The function $y = \sin(x)$ is one-to-one on the interval $-\pi/2 \leq x \leq \pi/2$. We will define the associated function $y = \text{Sin}(x)$ (shown in red on Figures 14.9(a) and (b) by restricting the domain of the sine function to $-\pi/2 \leq x \leq \pi/2$). On the given interval, we have $-1 \leq \text{Sin}(x) \leq 1$. We define the inverse function, called arcsine

$$y = \arcsin(x) \quad -1 \leq x \leq 1$$

in the usual way, by reflection of $\text{Sin}(x)$ through the line $y = x$ as shown in Figure 14.9(a).

To interpret this function, we note that $\arcsin(x)$ is “the angle whose sine is $x$”. In Figure 14.10, we show a triangle in which $\theta = \arcsin(x)$. This follows from the observation that the sine of theta, opposite over hypotenuse, is $x/1$ which is simply $x$. The length of the other side of the triangle is then $\sqrt{1-x^2}$ by the Pythagorean theorem.

For example $\arcsin(\sqrt{2}/2)$ is the angle whose sine is $\sqrt{2}/2$, namely $\pi/4$. (We see this by checking the values of trig functions of standard angles shown in Table 1.) A few other inter-conversions are given by the examples below.
14.3. Inverse Trigonometric functions

Figure 14.10. This triangle has been constructed so that $\theta$ is an angle whose sine is $x/1 = x$. This means that $\theta = \arcsin(x)$

The functions $\sin(x)$ and $\arcsin(x)$, reverse (or “invert”) each other’s effect, that is:

$$\arcsin(\sin(x)) = x \quad \text{for} \quad -\pi/2 \leq x \leq \pi/2,$$

$$\sin(\arcsin(x)) = x \quad \text{for} \quad -1 \leq x \leq 1.$$

There is a subtle point that the allowable values of $x$ that can be “plugged in” are not exactly the same for the two cases. In the first case, $x$ is an angle whose sine we compute first, and then reverse the procedure. In the second case, $x$ is a number whose arc-sine is an angle.

We can evaluate $\arcsin(\sin(x))$ for any value of $x$, but the result may not agree with the original value of $x$ unless we restrict attention to the interval $-\pi/2 \leq x \leq \pi/2$. For example, if $x = \pi$, then $\sin(x) = 0$ and $\arcsin(\sin(x)) = \arcsin(0) = 0$, which is not the same as $x = \pi$. For the other case, i.e. for $\sin(\arcsin(x))$, we cannot plug in any value of $x$ outside of $-1 \leq x \leq 1$, since $\arcsin(x)$ is simply not defined at all, outside this interval. This demonstrates that care must be taken in handling the inverse trigonometric functions.

**Inverse cosine**

We cannot use the same interval to restrict the cosine function, because it has the same $y$ values to the right and left of the origin. If we pick the interval $0 \leq x \leq \pi$, this difficulty is avoided, since we arrive at a one-to-one function. We will call the restricted-domain version of cosine by the name $y = \cos(x) = \cos(x)$ for $0 \leq x \leq \pi$. (See red curve in Figure 14.11(a). On the interval $0 \leq x \leq \pi$, we have $1 \geq \cos(x) \geq -1$ and we define the corresponding inverse function

$$y = \arccos(x) \quad -1 \leq x \leq 1$$

as shown in blue in Figure 14.11(b).

We understand the meaning of the expression $y = \arccos(x)$ as “the angle (in radians) whose cosine is $x$”. For example, $\arccos(0.5) = \pi/3$ because $\pi/3$ is an angle whose cosine is $1/2=0.5$. In Figure 14.12, we show a triangle constructed specifically so that $\theta = \arccos(x)$. Again, this follows from the fact that $\cos(\theta)$ is adjacent over hypotenuse. The length of the third side of the triangle is obtained using the Pythagorean theorem.

The inverse relationship between the functions mean that

$$\arccos(\cos(x)) = x \quad \text{for} \quad 0 \leq x \leq \pi,$$
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Figure 14.11. (a) The original function cos(x) is shown in black; the restricted domain version, Cos(x) is shown in red. The same red curve appears in part (b) on a slightly different scale. (b) Relationship between the functions Cos(x) (in red) and arccos(x) (in blue). Note that one is the reflection of the other about the line y = x.

Figure 14.12. This triangle has been constructed so that θ is an angle whose cosine is x/1 = x. This means that θ = arccos(x)

\[ \cos(\arccos(x)) = x \quad \text{for} \quad -1 \leq x \leq 1. \]

The same subtleties apply as in the previous case discussed for arc-sine.

**Inverse tangent**

The function \( y = \tan(x) \) is one-to-one on an interval \( \pi/2 < x < \pi/2 \). Unlike the case for \( \sin(x) \), we must exclude the endpoints, where the function \( \tan(x) \) is undefined. We therefore restrict the domain to \( \pi/2 < x < \pi/2 \), that is, we define,

\[ y = \Tan(x) = \tan(x) \quad \pi/2 < x < \pi/2. \]

Again in contrast to the sine function, as \( x \) approaches either endpoint of this interval, the value of \( \Tan(x) \) approaches \( \pm \infty \), i.e. \( -\infty < \Tan(x) < \infty \). This means that the domain
14.3. Inverse Trigonometric functions

Figure 14.13. (a) The function \( \tan(x) \), is shown in black, and \( \text{Tan}(x) \) in red. The same red curve is repeated in part (b) (b) Relationship between the functions \( \text{Tan}(x) \) (in red) and \( \text{arctan}(x) \) (in blue). Note that one is the reflection of the other about the line \( y = x \).

of the inverse function will be from \( -\infty \) to \( \infty \), i.e. the inverse function is defined for all real values of \( x \). We define the inverse tan function:

\[
y = \arctan(x) \quad -\infty < x < \infty.
\]

As before, we can understand the meaning of the inverse tan function, by constructing a triangle in which \( \theta = \arctan(x) \), shown in Figure 14.14.

\[
\sqrt{1+x^2}
\]

\[
\theta
\]

\[
x
\]

\[
l
\]

Figure 14.14. This triangle has been constructed so that \( \theta \) is an angle whose tan is \( x/l = x \). This means that \( \theta = \arctan(x) \)

The inverse tangent “inverts” the effect of the tangent on the relevant interval:

\[
\arctan(\tan(x)) = x \quad \text{for} \quad -\pi/2 < x < \pi/2
\]

\[
\tan(\arctan(x)) = x \quad \text{for} \quad -\infty < x < \infty
\]

The same comments hold in this case.
A summary of the above inverse trigonometric functions, showing their graphs on a single page is provided in Fig. F.3 in Appendix F. Some of the standard angles allow us to define precise values for the inverse trig functions. A table of such standard values is given in the same Appendix (See Table F.2). For other values of $x$, one has to calculate the decimal approximation of the function using a scientific calculator.

**Example 14.7** Simplify the following expressions: (a) $\arcsin(\sin(\pi/4))$, (b) $\arccos(\sin(-\pi/6))$

**Solution:** (a) $\arcsin(\sin(\pi/4)) = \pi/4$ since the functions are simple inverses of one another on the domain $-\pi/2 \leq x \leq \pi/2$.

(b) We evaluate this expression piece by piece: First, note that $\sin(-\pi/6) = -1/2$. Then $\arccos(\sin(-\pi/6)) = \arccos(-1/2) = 2\pi/3$. The last equality is obtained from Table F.2.

**Example 14.8** Simplify the expressions: (a) $\tan(\arcsin(x))$, (b) $\cos(\arctan(x))$.

**Solution:** (a) Consider first the expression $\arcsin(x)$, and note that this represents an angle (call it $\theta$) whose sine is $x$, i.e. $\sin(\theta) = x$. Refer to Figure 14.10 for a sketch of a triangle in which this relationship holds. Now note that $\tan(\theta)$ in this same triangle is the ratio of the opposite side to the adjacent side, i.e.

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}}$$

(b) Figure 14.14 shows a triangle that captures the relationship $\tan(\theta) = x$ or $\theta = \arctan(x)$. The cosine of this angle is the ratio of the adjacent side to the hypotenuse, so that

$$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2 + 1}}$$
Exercises

14.1. Convert the following expressions in radians to degrees:
   (a) \( \pi \) (b) \( 5\pi/3 \) (c) \( 21\pi/23 \) (d) \( 24\pi \)

Convert the following expressions in degrees to radians:
   (e) \( 100^\circ \) (f) \( 8^\circ \) (g) \( 450^\circ \) (h) \( 90^\circ \)

Using a Pythagorean triangle, evaluate each of the following:
   (i) \( \cos(\pi/3) \) (j) \( \sin(\pi/4) \) (k) \( \tan(\pi/6) \)

14.2. Graph the following functions over the indicated ranges:
   (a) \( y = x \sin(x) \) for \(-2\pi < x < 2\pi\)
   (b) \( y = e^x \cos(x) \) for \(0 < x < 4\pi\).

14.3. Sketch the graph for each of the following functions:
   (a) \( y = \frac{1}{2} \sin 3 \left( x - \frac{\pi}{4} \right) \)
   (b) \( y = 2 - \sin x \)
   (c) \( y = 3 \cos 2x \)
   (d) \( y = 2 \cos \left( \frac{1}{2} x + \frac{\pi}{4} \right) \)

14.4. The Radian is an important unit associated with angles. One revolution about a circle is equivalent to 360 degrees or \( 2\pi \) radians. Convert the following angles (in degrees) to angles in radians. (Express these as multiples of \( \pi \), not as decimal expansions):
   (a) 45 degrees
   (b) 30 degrees
   (c) 60 degrees
   (d) 270 degrees.

Find the sine and the cosine of each of these angles.

14.5. Find the appropriate trigonometric function to describe the following rhythmic processes:
   (a) Daily variations in the body temperature \( T(t) \) of an individual over a single day, with the maximum of \( 37.5^\circ \mathrm{C} \) at 8:00 am and a minimum of \( 36.7^\circ \mathrm{C} \) 12 hours later.
   (b) Sleep-wake cycles with peak wakefulness \( (W = 1) \) at 8:00 am and 8:00pm and peak sleepiness \( (W = 0) \) at 2:00pm and 2:00 am.

(For parts (a) and (b) express \( t \) as time in hours with \( t = 0 \) taken at 0:00 am.)

14.6. Find the appropriate trigonometric function to describe the following rhythmic processes:
   (a) The displacement \( S \) cm of a block on a spring from its equilibrium position, with a maximum displacement 3 cm and minimum displacement \(-3 \) cm, a period of \( \frac{2\pi}{\sqrt{g/l}} \) and at \( t = 0, S = 3 \).
(b) The vertical displacement $y$ of a boat that is rocking up and down on a lake. $y$ was measured relative to the bottom of the lake. It has a maximum displacement of 12 meters and a minimum of 8 meters, a period of 3 seconds, and an initial displacement of 11 meters when measurement was first started (i.e., $t = 0$).

14.7. **Sunspot cycles:** The number of sunspots (solar storms on the sun) fluctuates with roughly 11-year cycles with a high of 120 and a low of 0 sunspots detected. A peak of 120 sunspots was detected in the year 2000. Which of the following trigonometric functions could be used to approximate this cycle?

(A) $N = 60 + 120 \sin \left(\frac{2\pi}{11} (t - 2000) + \frac{\pi}{2}\right)$,  
(B) $N = 60 + 60 \sin \left(\frac{11}{2\pi} (t + 2000)\right)$

(C) $N = 60 + 60 \cos \left(\frac{11}{2\pi} (t + 2000)\right)$,  
(D) $N = 60 + 60 \sin \left(\frac{2\pi}{11} (t - 2000)\right)$

(E) $N = 60 + 60 \cos \left(\frac{2\pi}{11} (t - 2000)\right)$

14.8. **Inverse trigonometric functions:** The inverse trigonometric function $\arctan(x)$ (also written $\arctan(x)$) means the angle $\theta$ where $-\pi/2 < \theta < \pi/2$ whose tan is $x$. Thus $\cos(\arctan(x))$ (or $\cos(\arctan(x))$) is the cosine of that same angle. By using a right triangle whose sides have length 1, $x$ and $\sqrt{1+x^2}$ we can verify that

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}.$$

Use a similar geometric argument to arrive at a simplification of the following functions:

(a) $\sin(\arcsin(x))$,  
(b) $\tan(\arcsin(x))$,  
(c) $\sin(\arccos(x))$.

14.9. **Inverse trigonometric functions:** The value of $\tan(\arccos(x))$ is which of the following?

(A) $1 - x^2$,  
(B) $x$,  
(C) $1 + x^2$,  
(D) $\frac{\sqrt{1-x^2}}{x}$,  
(E) $\frac{\sqrt{1+x^2}}{x}$

14.10. **Inverse trigonometric functions:** The function $y = \tan(\arctan(x))$ has which of the following for its domain and range?

(A) Domain $0 \leq x \leq \pi$; Range $-\infty \leq y \leq \infty$  
(B) Domain $-\infty \leq x \leq \infty$; Range $-\infty \leq y \leq \infty$  
(C) Domain $-\pi \leq x \leq \pi$; Range $-\pi \leq y \leq \pi$;  
(D) Domain $-\pi/2 \leq x \leq \pi/2$; Range $-\pi/2 \leq y \leq \pi/2$;  
(E) Domain $-\infty \leq x \leq \infty$; Range $0 \leq y \leq \pi$
14.11. **Simplify:** Use a double-angle trigonometric identity to simplify the following expression as much as possible:

\[ y = \cos(2 \arcsin(x)). \]

For what values of \( x \) is this simplification possible?