Chapter 13

Qualitative methods for differential equations

Not all differential equations are easily solved analytically. Furthermore, even when we find the analytic solution, it is not always easy to interpret, graph, or understand. This motivates a number of qualitative methods that lead us to an overall understanding of the behaviour directly from information contained in the differential equation, without the challenges of finding a full functional form of the solution. In this chapter we will expand our familiarity with differential equations and assemble such new techniques for understanding these. We will expand our methods to deal with differential equations in which the expression on the left hand side (LHS) \( f(y) \) is nonlinear, i.e. equations of the form

\[
\frac{dy}{dt} = f(y)
\]

in which \( f \) is more complicated than the forms \( a - by \). Geometric techniques, rather than algebraic calculations will form the core of the concepts we will discuss.

13.1 Linear and nonlinear differential equations

Section 13.1 Learning goals

1. Understand the distinction between unlimited and density-dependent population growth. Be able to explain terms in the logistic equation in its original (13.1) and rescaled (13.3) versions.

2. Be able to state the definition of a linear differential equation.

3. Understand the law of mass action, and be able to derive simple differential equations for interacting species based on this law.
In our previous model for population growth, in Chapter 11, we encountered the differential equation

\[ \frac{dN}{dt} = kN, \]

where \( N(t) \) is population size at time \( t \) and \( k \) is a constant per capita growth rate. This differential equation, as we have seen, has exponential solutions. It means that two behaviours are generically obtained:\n
- Explosive growth if \( k > 0 \) or extinction if \( k < 0 \). But the \( k > 0 \) case is unrealistic. Most natural populations do not grow indefinitely in an explosive, exponential way. Eventually running out of space or resources, the population growth dwindles, and the population attains some static level rather than expanding forever. This motivates a revision of our previous model to depict density-dependent growth.

### 13.1.1 The logistic equation for population growth

Let \( N(t) \) represent the size of a population at time \( t \), as before. Consider the differential equation

\[ \frac{dN}{dt} = rN \left( \frac{K - N}{K} \right). \]  \hspace{1cm} (13.1)

We call this differential equation the **logistic equation**. Here the parameter \( r \) is the intrinsic growth rate and \( K \) is the carrying capacity. Both \( r, K \) are assumed to be positive constants for a given population in a given environment. The logistic equation has a long history in modelling population growth of microorganisms, animals, and human populations.

In the form written above, we could interpret the logistic equation as

\[ \frac{dN}{dt} = R(N) \cdot N, \quad \text{where } R(N) = \left[ r(K - N)/K \right] \]

then the term \( R(N) \), which replaces the constant rate of growth \( k \), is a so-called density dependent growth rate. We later show yet another interpretation that involves hostile interactions between individuals in the population.

### 13.1.2 Linear versus nonlinear

The logistic equation introduces the first example of a **nonlinear differential equation**. We explain the distinction and why it matters here.

**Definition 13.1 (Linear differential equation).** A first order differential equation is said to be linear if it is a linear combination of terms of the form

\[ \frac{dy}{dt}, \quad y, \quad 1 \]

that is, it can be written in the form

\[ \alpha \frac{dy}{dt} + \beta y + \gamma = 0 \]  \hspace{1cm} (13.2)

\[ ^{43} \text{There is also a third case, when } k = 0 \text{ in which neither growth nor decay occurs. But this case only happens when birth rate exactly equals mortality rate, so is hardly ever encountered.} \]
where $\alpha, \beta, \gamma$ do not depend on $y$. ("First order" means that only up to the first derivative occurs in the equation.)

So far, we have seen several examples of this type with constant coefficients $\alpha, \beta, \gamma$. For example, $\alpha = 1, \beta = -k, \gamma = 0$ in Eqn. 11.2 whereas $\alpha = 1, \gamma = -a, \beta = b$ in Eqn. (12.3). Any differential equation not of this simple form is said to be nonlinear.

**Example 13.2 (Linear versus nonlinear differential equations)** Which of the following differential equations are linear and which are nonlinear?

(a) $\frac{dy}{dt} = y^2$, (b) $\frac{dy}{dt} - y = 5$, (c) $y \frac{dy}{dt} = -1$.

**Solution:** Any term of the form $y^2, \sqrt{y}, 1/y$, etc. is nonlinear in $y$. A product such as $y \frac{dy}{dt}$ is also nonlinear in the independent variable. Hence equations (a), (c) are nonlinear, while (b) is linear.

The significance of the distinction between linear and nonlinear differential equations is that nonlinearities make it much harder to systematically find a solution to the given differential equation by “analytic” methods. Most linear differential equations have solutions that are made of exponential functions or expressions involving such functions. This is not true for nonlinear equations. However, as we will see shortly, geometric methods become very helpful in understanding the behaviour of such nonlinear differential equations.

### 13.1.3 Law of mass action

Nonlinear terms in differential equations arise in various ways. One common source is interactions between individuals that affect their state. Here is a simple example of this type.

Consider a chemical reaction in which molecules of type A bind to those of type B to react chemically and form some product P. Suppose we start out with a test-tube containing a mixture of A and B molecules at concentrations $a(t), b(t)$. These concentrations depend on time because the chemical reaction will use up both types in producing the product.

What can we say about the rate of the reaction? First, we note that the reaction only occurs when A and B molecules “collide” and stick to one another. Collisions occur randomly, but if the reaction vessel has more molecules of type A or more molecules of type B, then those collisions are more frequent, and hence the rate of reaction should be faster. This means that the rate of the reaction should depend on both chemical concentrations $a(t)$ and $b(t)$. The simplest assumption that captures this dependence is

$$ \text{rate of reaction} \sim a \cdot b \Rightarrow \text{rate of reaction} = k \cdot a \cdot b $$

where $k$ is some constant that represents the reactivity of the molecules.

We can formally state this result, known as the **Law of Mass Action** as follows:
The Law of Mass Action: The rate of a chemical reaction involving an interaction of two or more chemical species is proportional to the product of the concentrations of the given species.

Example 13.3 (Differential equation for interacting chemicals) In a 1 litre chemical reactor, substance A is added at a constant rate of \( I \) moles per hour. There, pairs of molecules of A interact chemically to form some product. Assuming that the volume does not change, write down a differential equation that keeps track of concentration of A in the reactor, \( y(t) \).

Solution: First suppose that there is no reaction. Then the addition of A to the reactor at a constant rate would lead to changing \( y(t) \), which would satisfy the differential equation

\[
\frac{dy}{dt} = I.
\]

When the chemical reaction takes place, there is a depletion of A which depends on interaction of pairs of molecules. But according to the law of mass action, such a term would be of the form \( k \cdot y \cdot y = ky^2 \). This reduces the concentration of \( a \), so it contributes to a negative rate of change. Hence

\[
\frac{dy}{dt} = I - ky^2.
\]

This is a nonlinear differential equation, as it contains a term of the form \( y^2 \).

Example 13.4 (Logistic equation reinterpreted) Rewrite the logistic equation in the form

\[
\frac{dN}{dt} = rN - bN^2
\]

(where \( b = r/K \) is a positive quantity). Interpret the meaning of this restated form of the equation by explaining what each of the terms on the right hand side could represent. Which of the two terms would be most significant for small versus for large population levels?

Solution: This form of the equation has growth term \( rN \) proportional to population size, which we have encountered before in unlimited population growth. However, there is also a quadratic (nonlinear) rate of loss (note minus sign) \(-bN^2\). This term could describe interactions between individuals that lead to mortality, e.g. through fighting or competition. From familiarity with power functions (in this case, the functions of \( N \) that form the two terms, \( rN \) and \( bN^2 \)) we can deduce that the second, quadratic term will dominate for larger values of \( N \), and this means that when the population is crowded, the loss of individuals is greater than the rate of reproduction.
13.1.4 Scaling the logistic equation

The logistic equation (13.1) involves two parameters, \( r \) and \( K \). Since units on each side of an equation must balance, and must be the same for terms that are added or subtracted, we can infer that \( K \) has the same units as \( N \), and indeed, that it is a population density. When \( N = K \), the population growth rate is zero \( (dN/dt = 0) \).

It turns out that we can understand the behaviour of the logistic equation by converting it to a “generic” form that does not depend on the constant \( K \). We do so by transforming variables, which amounts to choosing a convenient way to measure the population size.

Example 13.5 (Rescaling:) Define a new variable

\[
y(t) = \frac{N(t)}{K}.
\]

Then \( N(t) = Ky(t) \). Interpret what the transformed variable \( y \) represents, and rewrite the logistic equation in terms this variable.

Solution: The variable, \( y(t) \) represents a scaled version of the population density. Instead of measuring the population in some arbitrary units such as number of individuals per acre, or number of bacteria per ml, \( y(t) \) measures the population in “multiples of the carrying capacity”. For example, if the environment can sustain 1000 aphids per plant (so \( K = 1000 \) individuals per plant), and the current population size on a given plant is \( N = 950 \) then the value of the scaled variable is \( y = 950/1000 = 0.95 \). (We would say that “the aphid population is at 95% of its carrying capacity on the plant.”) Since \( K \) is assumed constant, it follows that

\[
N(t) = Ky(t) \quad \Rightarrow \quad \frac{dN}{dt} = K \frac{dy}{dt}.
\]

Using this, we can simplify the logistic equation as follows:

\[
\frac{dN}{dt} = rN\frac{(K - N)}{K} , \quad \Rightarrow \quad K \frac{dy}{dt} = r(Ky)\frac{(K - Ky)}{K} , \quad \Rightarrow \quad \frac{dy}{dt} = ry(1 - y).
\]

Equation (13.3) “looks simpler” than (13.1) since it depends on only one parameter, \( r \). Moreover, by understanding this equation, and transforming back to the original logistic in terms of \( N(t) = Ky(t) \), we can interpret results for the original model. While we do not go further with transforming variables at present, it turns out that one can also reduce the scaled logistic to an equation in which \( r = 1 \) by “rescaling time units”. This is left as an optional exercise for the reader.

13.2 The geometry of change

In this section, we introduce a new method for understanding differential equations using graphical and geometric arguments. Such methods circumvent the solutions that we expressed in terms of analytic formulae. We resort to concepts learned much earlier in this course: the derivative as a slope of a tangent line, in order to use the differential equation itself to assemble a sketch of the behaviour that it predicts. That is, rather than writing
down \( y = F(t) \) as a solution to the differential equation (and then graphing that function) we sketch the qualitative behaviour of such solution curves directly from information contained in the differential equation.

### Section 13.2 Learning goals

1. Understand the idea of a **slope field** of a differential equation. Given a differential equation (linear or nonlinear), be able to construct such a diagram and use it to sketch solution curves.

2. Understand the idea of a **state-space diagram**, and be able to construct such a diagram and use it to interpret the behaviour of solution curves to a given differential equation.

3. Understand the relationships between a slope field, a state-space diagram, and a family of solution curves to a given differential equation.

4. Be able to identify steady states of a differential equation and determine whether they are stable or unstable.

5. Given a differential equation and initial condition, be able to predict the behaviour of the solution for \( t > 0 \).

#### 13.2.1 Slope fields

Here we discuss a geometric way of understanding what a differential equation is saying using a **slope field**, also denoted **direction field**. We have already seen that solutions to a differential equation of the form

\[
\frac{dy}{dt} = f(y)
\]

are curves in the \( y, t \) plane that describe how \( y(t) \) changes over time. (Thus, these curves are graphs of functions of time.) Each initial condition \( y(0) = y_0 \) is associated with one of these curves, so that together, these curves form a family of solutions. What do these curves have in common geometrically?

Simply stated, the slope of the tangent line (which is just \( dy/dt \)) at any point on any of the curves has to be related to the value of the \( y \) coordinate of that point as stated in the differential equation. That is exactly what the differential equation is saying: at any point \( (t, y(t)) \) on a solution curve, the tangent line must have slope \( f(y) \), which depends only on the \( y \) value\(^4\), and not on the time \( t \). By sketching slopes at various values of \( y \), we obtain the slope field from which we can get a reasonable idea of the behaviour of the solutions to the differential equation.

\(^4\)In more general cases, the expression \( f(y) \) that appears in the differential equation might depend on \( t \) as well as \( y \). For the purpose of this course, we will not consider such examples in detail.
Example 13.6 Consider the differential equation

\[ \frac{dy}{dt} = 2y. \]  

(13.4)

Compute some of the slopes for various \( y \) values and use this to sketch a **slope field** for this differential equation.

Solution: Equation (13.4) states that if a solution curve passes through a point \((t, y)\), then its tangent line at that point has a slope \(2y\), regardless of the value of \(t\). This example is simple enough that we can state the following: for positive values of \(y\), the slope is positive, for negative values of \(y\), the slope is negative, and for \(y = 0\) the slope is zero. We provide some tabulated values of \(y\) indicating the values of the slope \(f(y)\), its sign, and what this implies about the local behaviour of the solution and its direction. Then, in Figure 13.1 we combine this information to generate the direction field and the corresponding solution curves. Note that the direction of the arrows (rather than their absolute magnitude) provides the most important qualitative tendency for the slope field sketch.

<table>
<thead>
<tr>
<th>(y)</th>
<th>(f(y) = 2y)</th>
<th>slope of tangent line</th>
<th>behaviour of (y)</th>
<th>direction of arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-4</td>
<td>-ve</td>
<td>decreasing</td>
<td>↘</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>-ve</td>
<td>decreasing</td>
<td>↘</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>no change in (y)</td>
<td>→</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>+ve</td>
<td>increasing</td>
<td>↗</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>+ve</td>
<td>increasing</td>
<td>↗</td>
</tr>
</tbody>
</table>

Table 13.1. **Table of derivatives and slopes for the differential equation** (13.4) in Example 13.6.

![Slope field and solution curves](image)

In constructing the slope field and solution curves, the following basic rules should be followed:
1. By convention, time flows from left to right along the \( t \) axis in our graphs, so the direction of all arrows (not indicated explicitly on the slope field) is always from left to right.

2. According to the differential equation, for any given value of the variable \( y \), the slope is given by the expression \( f(y) \) in the differential equation. The sign of that quantity is particularly important in determining whether the solution is locally increasing, decreasing, or neither. In the tables, we indicate this in the last column with the notation \( \nearrow, \searrow \), or \( \rightarrow \).

3. There is a single arrow at any point in the ty plane, and consequently solution curves cannot intersect anywhere (although they can get arbitrarily close to one another).

We will see some implications of these rules in our examples.

**Example 13.7** Consider the differential equation

\[
\frac{dy}{dt} = f(y) = y - y^3.
\]

Create a slope field diagram for this differential equation.

<table>
<thead>
<tr>
<th>( y )</th>
<th>sign of ( f(y) = y - y^3 )</th>
<th>behaviour of ( y )</th>
<th>direction of arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y &lt; -1 )</td>
<td>+ve</td>
<td>increasing</td>
<td>( \nearrow )</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>no change in ( y )</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>-0.5</td>
<td>-ve</td>
<td>decreasing</td>
<td>( \searrow )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>no change in ( y )</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>0.5</td>
<td>+ve</td>
<td>increasing</td>
<td>( \nearrow )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>no change in ( y )</td>
<td>( \rightarrow )</td>
</tr>
<tr>
<td>( y &gt; 1 )</td>
<td>-ve</td>
<td>decreasing</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

**Table 13.2. Table for Example 13.7.**

**Solution:** Based on the last example, we will focus on the sign, rather than the value of the derivative \( f(y) \), since that sign determines whether the solutions increase, decrease, or stay constant. Factoring an expression usually helps to find its zeros, and then to identify where it changes sign. For example,

\[
\frac{dy}{dt} = f(y) = y - y^3 = y(1 - y^2) = y(1 + y)(1 - y).
\]

The sign of \( f \) depends on the signs of the factors \( y, (1 + y), (1 - y) \). For \( y < -1 \), two factors, \( y, (1 + y) \), are negative, whereas \( (1 - y) \) is positive, so that the product is positive overall. The sign of \( f(y) \) changes at each of the three points \( y = 0, \pm 1 \) where one or another of the three factors changes sign, as shown in Table 13.2. Eventually, to the right of all three (when \( y > 1 \)), the sign is negative. We summarize these observations in Table 13.2 and show the slopes field and solution curves in Fig 13.2.
Example 13.8 Sketch a slope field and solution curves for the problem of a cooling object, and specifically for

$$\frac{dT}{dt} = f(T) = 0.2(10 - T), \quad \text{(13.5)}$$

<table>
<thead>
<tr>
<th>$T$</th>
<th>sign of $f(T) = 0.2(10 - T)$</th>
<th>behaviour of $T$</th>
<th>direction of arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T &lt; 10$</td>
<td>+ve</td>
<td>increasing</td>
<td>↗</td>
</tr>
<tr>
<td>$T = 10$</td>
<td>0</td>
<td>no change</td>
<td>→</td>
</tr>
<tr>
<td>$T &gt; 10$</td>
<td>-ve</td>
<td>decreasing</td>
<td>↘</td>
</tr>
</tbody>
</table>

Table 13.3. Slopes for Example 13.8.

Solution: The family of curves shown in Figure 12.2 are solutions for the temperature. The function $f(T) = 0.2(10 - T)$ corresponds to the slopes of tangent lines to these curves, and we indicate the sign of $f(T)$ and thereby the behaviour of $T(t)$ in Table 13.3. Note that there is only one change of sign, at $T = 10$. (For smaller $T$, the solution is always increasing and for larger $T$, the solution is always decreasing.) The slope field is then shown in Figure 13.3(a) with solution curves in (b). In this example, we see that the detailed solutions found analytically (Example 12.6), found using Euler’s method (Example 12.13), and those sketched using the new qualitative arguments agree.
13.2.2 State-space diagrams

In Examples 13.6-13.8, we have already seen that we can understand qualitative features of solutions to the differential equation

\[ \frac{dy}{dt} = f(y) \]  

(13.6)

by examining the expression \( f(y) \). Up to now, we have used the sign of \( f(y) \) to assemble a slope field diagram and sketch solution curves. The slope field informed us about which initial values of \( y \) would increase, decrease or stay constant. We next show another way of determining the same information. First, let us define a state space, also called phase line, which is essentially the \( y \) axis with superimposed arrows representing the direction of flow.

**Definition 13.9 (State space (or Phase line)).** A line representing the dependent variable \( (y) \) together with arrows to describe the flow along that line (increasing, decreasing, or stationary \( y \)) satisfying (13.6) is called the state space diagram or the phase line diagram for the differential equation.

Rather than tabulating signs for \( f(y) \), we could arrive at similar conclusions by sketching \( f(y) \) and observing where this function is positive (implying that \( y \) increases) or negative (\( y \) decreases). Places where \( f(y) = 0 \) (“zeros of \( f \”) are important boundaries between such regimes and also important in their own right for signifying “static solutions” (no change in \( y \)). Along the \( y \) axis (which is now on the horizontal axis of the sketch) increasing \( y \) means motion to the right, decreasing \( y \) means motion to the left.

As we shall see, the information contained in this type of diagram provides qualitative description of solutions to the differential equation, but with the explicit time behaviour suppressed. This is illustrated by Fig. 13.4, where we show the connection between the slope field diagram and the state space diagram for a typical differential equation.
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Figure 13.4. The relationship of the slope field and state space diagrams. (a) A typical slope field. A few arrows have been added to indicate the direction of time flow along the tangent vectors. Now consider “looking down the time axis” as shown by the “eye” in this diagram. Then the t axis points towards us, and we see only the y axis as in (b). Arrows on the y axis indicate the directions of flow for various values of y as determined in (a). Now “rotate” the y axis so it is horizontal, as shown in (c). The direction of the arrows exactly correspond to places where \( f(y) \), in (c), is positive (which implies increasing y, \( \to \)), or negative (which implies decreasing y, \( \leftarrow \)). The state space diagram is the y axis in (b) or (c).

Example 13.10 Consider the differential equation

\[
\frac{dy}{dt} = f(y) = y - y^3. \tag{13.7}
\]

Sketch \( f(y) \) versus y and use your sketch to determine where y is static, and where y increases or decreases. Then describe in words what this predicts starting from each of the three initial conditions: (i) \( y(0) = -0.5 \), (ii) \( y(0) = 0.3 \), or (iii) \( y(0) = 2 \).

Solution: From a previous example, we know that \( f(y) = 0 \) at for \( y = -1, 0, 1 \). This means that y does not change at these values, so, if we start a system off with \( y(0) = 0 \), or \( y(0) = \pm 1 \), the value of y will be static. The three places at which this happens are marked by heavy dots in Figure 13.5(a).

We also see that \( f(y) < 0 \) for \(-1 < y < 0\) and for \( y > 1 \). In these intervals, \( y(t) \) must be a decreasing function of time (\( dy/dt < 0 \)). On the other hand, for \( 0 < y < 1 \) or for \( y < -1 \), we have \( f(y) > 0 \), so \( y(t) \) is increasing. See arrows on Figure 13.5(b). We see from this figure that there is a tendency for y to move away from the value \( y = 0 \) and to approach either of the values 1 or -1. Starting from the initial values given above, we have (i) \( y(0) = -0.5 \) results in \( y \to -1 \), (ii) \( y(0) = 0.3 \) leads to \( y \to 1 \), and (iii) \( y(0) = 2 \) implies \( y \to \infty \).

Example 13.11 (A cooling object:) Sketch the same type of diagrams for the problem of a cooling object and interpret its meaning.
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Figure 13.5. Static points (dots) and intervals for which \( y \) increases or decreases for the differential equation (13.7). See Example 13.10.

Solution: Here, the differential equation is

\[
\frac{dT}{dt} = f(T) = 0.2(10 - T).
\]

A sketch of the rate of change, \( f(T) \) versus the temperature \( T \) is shown in Figure 13.6(a). We deduce the direction of the flow directly from this sketch.

Example 13.12 Create a similar qualitative sketch for the more general form of linear differential equation

\[
\frac{dy}{dt} = f(y) = a - by.
\]  

(13.8)

For what values of \( y \) would there be no change?

Solution: The rate of change of \( y \) is given by the function \( f(y) = a - by \). This is shown in the sketch in Figure 13.6(b). We see that there is one point at which \( f(y) = 0 \), namely at
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\[ y = \frac{a}{b}. \] Starting from an initial condition \( y(0) = \frac{a}{b} \), there would be no change. We also see from this figure that \( y \) approaches this value over time. After a long time, the value of \( y \) will be approximately \( \frac{a}{b} \).

### 13.2.3 Steady states and stability

We notice from Figure 13.3 that for a certain initial temperature, namely \( T_0 = 10 \), there will be no change with time. Indeed, we find that at this temperature the differential equation specifies that \( \frac{dT}{dt} = 0 \). Such a value is called a **steady state**.

**Definition 13.13 (Steady state).** A **steady state** is a state in which a system is not changing.

**Example 13.14** Find the steady states of the equation (13.7).

**Solution:** To find steady states we look for \( y \) such that \( \frac{dy}{dt} = 0 \). But these are just points that satisfy \( f(y) = 0 \), that is zeros of \( f \). Thus \( y = 0 \) and \( y = \pm 1 \) are the three steady states of this differential equation.

From Figure 13.5, we see that solutions starting close to \( y = 1 \) tend to get closer and closer to this value. We refer to this behaviour as **stability** of the steady state.

**Definition 13.15 (Stability).** We say that a steady state is **stable** if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is **unstable**, if states that are initially very close to it eventually move away from that steady state.

**Example 13.16** Determine the stability properties of steady states of Eqn. (13.7).

**Solution:** From any starting value of \( y > 0 \) in this example, we see that after a long time, the solution curves tend to approach the value \( y = 1 \). States close to \( y = 1 \) get closer to it, so this is a stable steady state. For the steady state \( y = 0 \), we see that initial conditions close to \( y = 0 \) do not get closer, but rather move away over time. Thus, this steady state is unstable. Similarly, the steady state at \( y = -1 \) is stable. In Fig. 13.5 we show the stable steady states with black dots and the unstable steady state with an open dot.

### 13.3 Applying qualitative analysis to biological models

The qualitative and geometric ideas developed in this chapter will help us to understand a variety of differential equations that stem from biological, physical, or chemical applications. In the following sections we will first use the methods to obtain a thorough understanding of **logistic population growth**. We then derive a model for the spread of a disease, and use similar qualitative arguments to analyze the predictions of that model.
Section 13.3 Learning goals

1. Practice the techniques of slope field, state-space diagram, and steady state analysis on the logistic equation.

2. Follow the derivation of a model for interacting (healthy, infected) individuals based on a set of assumptions.

3. Understand that the resulting set of two ODEs can be reduced to a single ODE. Be able to use qualitative methods to analyse the model behaviour and to interpret the results.

13.3.1 Qualitative analysis of the logistic equation

We apply the new methods to the logistic equation.

Example 13.17 Find the steady states of the logistic equation (13.1).

Solution: To determine the steady states of the equation (13.1), i.e. the level of population that would not change over time, we look for values of \( N \) such that

\[
\frac{dN}{dt} = 0.
\]

This leads to

\[
rN\left(\frac{K - N}{K}\right) = 0,
\]

which has solutions \( N = 0 \) (no population at all) or \( N = K \) (the population is at its carrying capacity). We could similarly find steady states of the scaled form of the logistic equation, (13.3). Setting \( \frac{dy}{dt} = 0 \) leads to

\[
0 = \frac{dy}{dt} = ry(1 - y) \quad \Rightarrow \quad y = 0, \text{ or } y = 1.
\]

This comes as no surprise since these values of \( y \) correspond to the values \( N = 0 \) and \( N = K \).

Example 13.18 Draw a plot of the rate of change \( \frac{dy}{dt} \) versus the value of \( y \) for the scaled logistic equation (13.3).

Solution: This plot is shown in Figure 13.7. In the interval \( 0 < y < 1 \), the rate of change is positive, so that \( y \) increases, whereas for \( y > 1 \), the rate of change is negative, so \( y \) decreases. Since \( y \) refers to population size, we need not concern ourselves with behaviour for \( y < 0 \). From Figure 13.7 we deduce that solutions that start with a positive \( y \) value will all approach \( y = 1 \) with time. (Solutions starting at \( y = 0 \) or \( y = 1 \) would not change.) Restated in terms of the variable \( N(t) \), any initial population should approach its carrying capacity \( K \) with time. We now look at the same equation from the perspective of the slope field.
13.3. Applying qualitative analysis to biological models

Rate of change

\[
\frac{dy}{dt}
\]

\[y\]

Figure 13.7. *Plot of \(\frac{dy}{dt}\) versus \(y\) for the the scaled logistic equation (13.3).*

**Example 13.19** Draw a slope field for the scaled logistic equation with \(r = 0.5\), that is for

\[
\frac{dy}{dt} = f(y) = 0.5 \cdot y(1 - y).
\]  

(13.9)

**Solution:** We generate slopes for various values of \(y\) in Table 13.4 and plot the slope field in Figure 13.8(a).

<table>
<thead>
<tr>
<th>(y)</th>
<th>sign of (f(y) = 0.5y(1 - y))</th>
<th>behaviour of (y)</th>
<th>direction of arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>no change in (y)</td>
<td>→</td>
</tr>
<tr>
<td>0 &lt; (y) &lt; 1</td>
<td>+ve</td>
<td>increasing</td>
<td>(\uparrow)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>no change in (y)</td>
<td>→</td>
</tr>
<tr>
<td>(y &gt; 1)</td>
<td>-ve</td>
<td>decreasing</td>
<td>(\downarrow)</td>
</tr>
</tbody>
</table>

Table 13.4. *Table for slope field for the logistic equation (13.9). See Fig 13.8(a) for the resulting diagram.*

Finally, we practice Euler’s method to graph the numerical solution to (13.9) from several initial conditions.

**Example 13.20 (Numerical solutions to the logistic equation):** Use Euler’s method to approximate the solutions to the logistic equation (13.9).

**Solution:** In Figure 13.8(b) we show a set of solution curves, obtained by solving the equation numerically using Euler’s method and the spreadsheet. To obtain these solutions, a value of \(h = \Delta t = 0.1\) was used. The solution is plotted for each initial condition.
Chapter 13. Qualitative methods for differential equations

Figure 13.8. (a) Slope field and (b) solution curves for the logistic equation (13.9).

\[ y(0) = y_0. \] The successive values of \( y \) were calculated according to

\[
\begin{align*}
y_1 &= y_0 + 0.5y_0(1 - y_0)h \\
y_2 &= y_1 + 0.5y_1(1 - y_1)h \\
&\vdots \\
y_{k+1} &= y_k + 0.5y_k(1 - y_k)h.
\end{align*}
\]

From Fig. 13.8, we see that solution curves approach \( y = 1 \). This means that the population \( N(t) \) approaches the carrying capacity \( K \) for all nonzero starting values, i.e. there will be a stable steady state with a fixed level of the population.

These results were generated on a spreadsheet. A useful feature of spreadsheets is that the repetitive calculations can be handled automatically by dragging a cell entry containing the results for one iteration down to generate other iterations. Once the method is implemented, it is also easy to change the initial condition, just by changing a single cell entry.

Example 13.21 (Inflection points) Some of the curves shown in Figure 13.8(b) have an inflection point, but others do not. Use the differential equation to determine which of the solution curves will have an inflection point.

Solution: From Figure 13.8(b) we might observe that the curves that emanate from initial values in the range \( 0 < y_0 < 1 \) are all increasing. Indeed, this follows from the fact that if \( y \) is in this range, the rate of change \( ry(1 - y) \) is a positive quantity.

The logistic equation has the form

\[
\frac{dy}{dt} = ry(1 - y) = ry - ry^2
\]

This means that (by differentiating both sides and remembering the chain rule)

\[
\frac{d^2y}{dt^2} = r \frac{dy}{dt} - 2ry \frac{dy}{dt} = r \frac{dy}{dt}(1 - 2y).
\]
An inflection point would occur at places where the second derivative changes sign, and in addition
\[
\frac{d^2y}{dt^2} = 0.
\]
From the above we see that this is possible for \(dy/dt = 0\) or for \((1 - 2y) = 0\). We have already dismissed the first possibility because we have argued that the rate of change in nonzero in the interval of interest. Thus we conclude that an inflection point would occur whenever \(y = 1/2\). Any initial condition satisfying \(0 < y_0 < 1/2\) would eventually pass through \(y = 1/2\) on its way up to the steady state level at \(y = 1\), and in so doing, would have an inflection point.

### 13.3.2 A model for the spread of a disease

In the era of human immunodeficiency virus (HIV), Severe Acute Respiratory Syndrome (SARS), Avian influenza (“bird flu”) and similar emerging infectious diseases, it is prudent to consider how infection spreads, and how it could be controlled or suppressed. Sustaining public health requires an understanding of the dynamics of emergent diseases, and the factors that determine whether they become endemic, epidemic, or of limited danger to human health. This motivates the discussion in this section.

For simplicity, we restrict attention to a non-lethal disease (such as the common cold or mild influenza). We consider a population in which some individuals are healthy and some are infected. We will assume that all healthy individuals are susceptible to catching the infection, and those that are currently infected are also infectious, and can transmit the infection to others through social interactions. We also assume that the infection is mild enough that individuals recover at some constant rate, and that they become susceptible once recovered. \(^{45}\) Furthermore, we will consider this scenario in the context of a fixed population (with no birth, death or migration during the timescale of interest). Our goal in this section is to predict whether the infection would spread and take hold in the population or whether it would run its course and disappear. We will find that this example illustrates the methods used so far and allows us to draw conclusions that were not intuitively obvious to begin with.

Let us use the following notation:

\[
S(t) = \text{size of population of susceptible (healthy) individuals,}
\]
\[
I(t) = \text{size of population of infected individuals,}
\]
\[
N(t) = S(t) + I(t) = \text{total population size.}
\]

We add a few simplifying assumptions.

1. The population mixes very well, so each individual is equally likely to contact and interact with any other individual. The contact is random.

2. Other than the state \((S \text{ or } I)\), individuals are “identical”. They recover at the same (constant) rate, and they have the same tendency to become infected.

\(^{45}\) Usually, recovery from an illness leads to partial temporary immunity. While this, too, can be modelled, we restrict attention to the simpler case which is tractable using mathematics we have just introduced.
3. On the timescale of interest, there is no birth, death or migration, only exchange between $S$ and $I$.

**Example 13.22** Suppose that the process can be represented by the scheme

\[
S + I \rightarrow I + I,
\]
\[
I \rightarrow S
\]

The first part, transmission of disease from $I$ to $S$ involves interaction. The second part is recovery. Use the assumptions above to track the two populations and to formulate a set of differential equations for $I(t)$ and $S(t)$.

**Solution:** We first write down the following “word equations” to keep track of individuals

\[
\begin{bmatrix}
\text{Rate of change of } I(t)
\end{bmatrix} = \begin{bmatrix}
\text{Rate of Gain due to disease transmission}
\end{bmatrix} - \begin{bmatrix}
\text{Rate of loss due to recovery}
\end{bmatrix}
\]

According to our assumption, recovery takes place at a constant rate. We denote that rate by $\mu > 0$ per unit time. By the law of mass action, the disease transmission rate should be proportional to the product of the populations, $(S \cdot I)$. Assigning $\beta > 0$ to be the constant of proportionality leads to the following differential equations for the infected population (which simply restates the “word equation” in mathematical notation):

\[
\frac{dI}{dt} = \beta SI - \mu I.
\]

Similarly, we can write a word equation that tracks the population of susceptibles:

\[
\begin{bmatrix}
\text{Rate of change of } S(t)
\end{bmatrix} = -\begin{bmatrix}
\text{Rate of Loss due to disease transmission}
\end{bmatrix} + \begin{bmatrix}
\text{Rate of gain due to recovery}
\end{bmatrix}
\]

Observe that loss from one group leads to (exactly balanced) gain in the other group. By similar logic, the differential equation for $S(t)$ is then

\[
\frac{dS}{dt} = -\beta SI + \mu I.
\]

We have arrived at two differential equations that describe the changes in each of the groups,

\[
\frac{dI}{dt} = \beta SI - \mu I, \quad (13.10a)
\]
\[
\frac{dS}{dt} = -\beta SI + \mu I. \quad (13.10b)
\]

It is evident from Eqs. (13.10) that changes in one population are linked to the levels of both, which means that the differential equations are **coupled** (linked to one another). Hence, we cannot “solve one” independently of the other. We must treat them as a pair. However, as we will observe in the next examples, we can simplify this system of equations using the fact that the total population does not change.
13.3. Applying qualitative analysis to biological models

Example 13.23 Use equations (13.10) to show that the total population does not change. (Hint: show that the derivative of \( S(t) + I(t) \) is zero.)

Solution: Add the equations to one another. Then we obtain

\[
\frac{d}{dt} [I(t) + S(t)] = \frac{dI}{dt} + \frac{dS}{dt} = \beta SI - \mu I - \beta SI + \mu I = 0.
\]

Hence

\[
\frac{d}{dt} [I(t) + S(t)] = \frac{dN}{dt} = 0,
\]

which means that the total population does not change, so that \( N(t) = [I(t) + S(t)] = N=\text{constant}. \) In contrast to the logistic equation (13.1), here \( N \) is a constant and \( I(t), S(t) \) are the variables.

Example 13.24 Use the fact that \( N \) is constant to express \( S(t) \) in terms of \( I(t) \) and \( N \), and eliminate \( S(t) \) from the differential equation for \( I(t) \). Your equation will contain only the constants \( N, \beta, \mu \).

Solution: Since \( N = S(t) + I(t) \) is constant, we can write \( S(t) = N - I(t) \). Then, plugging this into the differential equation for \( I(t) \) we obtain

\[
\frac{dI}{dt} = \beta SI - \mu I, \quad \Rightarrow \quad \frac{dI}{dt} = \beta(N - I)I - \mu I.
\]

Example 13.25 Show that the above equation can be written in the form

\[
\frac{dI}{dt} = \beta I(K - I),
\]

where \( K \) is a constant, and determine how this constant depends on \( N, \beta, \mu \). Is the constant \( K \) positive or negative?

Solution: We rewrite the differential equation for \( I(t) \) as follows:

\[
\frac{dI}{dt} = \beta(N - I)I - \mu I = \beta I \left( N - I - \frac{\mu}{\beta} \right) = \beta I \left( N - \frac{\mu}{\beta} - I \right).
\]

Then, we identify the constant,

\[
K = \left( N - \frac{\mu}{\beta} \right).
\]

Evidently, \( K \) could be either positive or negative, that is

\[
\begin{cases} 
N \geq \frac{\mu}{\beta} & \Rightarrow K \geq 0, \\
N < \frac{\mu}{\beta} & \Rightarrow K < 0.
\end{cases}
\]

Using the above process, we have reduced the system of two differential equations for the two variables \( I(t), S(t) \) to a single differential equation for \( I(t) \), together with the statement \( S(t) = N - I(t) \). We now examine implications of this result using qualitative methods developed in this chapter.
Example 13.26 Consider the differential equation for $I(t)$ given by

$$\frac{dI}{dt} = \beta I(K - I), \quad \text{where} \quad K = \left( N - \frac{\mu}{\beta} \right). \quad (13.11)$$

Find the steady states of the differential equation (13.11) and draw a state space diagram in each of the following two cases: (a) $K \geq 0$, (b) $K < 0$. Use your diagram to determine which steady state(s) are stable or unstable.

Solution: Steady states of Eqn. (13.11) satisfy $dI/dt = 0$, namely $\beta I(K - I) = 0$. The possible roots are $I = 0$ (no infected individuals) and $I = K$. The latter can only make sense if $K \geq 0$. We plot the function $f(I) = \beta I(K - I)$ in Eqn. (13.11) against the state variable $I$ in both cases. Observe that this function is quadratic in $I$, and, as in the logistic equation, its graph is a parabola opening downwards. We add arrows pointing right ($\rightarrow$) in the regions where $dI/dt > 0$ and arrows pointing left ($\leftarrow$) where $dI/dt < 0$. In case (a), when $K \geq 0$, we find that arrows point towards $I = K$, so this steady state is stable. Arrows point away from $I = 0$, so this represents an unstable steady state. In case (b), while we still have a parabolic graph with two steady states, the state $I = K$ is not admissible since $K$ is negative. Hence only one steady state, at $I = 0$ is relevant biologically, and all initial conditions will move towards this state.

Example 13.27 Interpret the results of the model in terms of the disease, assuming that initially most of the population is in the $S$ group, and a small number of infected individuals are present at $t = 0$.

Solution: In case (a), as long as the initial size of the infected group is positive ($I > 0$), with time it will approach $K$, that is, $I(t) \to K = N - \mu/\beta$. The rest of the population will be in the susceptible group, that is $S(t) \to \mu/\beta$ (so that $S(t) + I(t) = N$ is always constant.) This first scenario holds provided $K > 0$ which is equivalent to $N > \mu/\beta$. There
will then be some infected and some healthy individuals in the population indefinitely, according to the model. In this case, we say that the disease becomes **endemic**.

In case (b), which corresponds to $N < \mu / \beta$, we see that $I(t) \to 0$ regardless of the initial size of the infected group. In that case, $S(t) \to N$ so with time, the infected group will shrink and the healthy group will grow so that the whole population becomes healthy. From these two results, we can conclude that the disease will be wiped out in a small population, whereas in a large population, it will spread until a steady state is attained where some fraction of the population is infected. In fact we have identified a threshold that separates these two behaviours:

\[
\frac{N\beta}{\mu} > 1 \Rightarrow \text{disease becomes endemic,}
\]

\[
\frac{N\beta}{\mu} < 1 \Rightarrow \text{disease is wiped out.}
\]

The ratio of constants in these inequalities, $R_0 = N\beta / \mu$ is called the **basic reproductive number** for the disease. Many current and much more detailed models for disease transmission also have threshold behaviour, and the ratio that determines whether the disease spreads or disappears, $R_0$ is of great interest in vaccination strategies. This ratio represents the number of infections that arise when 1 infected individual interacts with a population of $N$ susceptible individuals.
Exercises

13.1. Explain the connection between Eqn. (13.2) and the equations Eqn. 11.2 and Eqn. (12.3).

13.2. **Slope fields:** Consider the differential equations given below. In each case, draw a slope field, determine the values of \( y \) for which no change takes place [such values are called steady states] and use your slope field to predict what would happen starting from an initial value \( y(0) = 1 \).

- (a) \( \frac{dy}{dt} = -0.5y \)
- (b) \( \frac{dy}{dt} = 0.5y(2 - y) \)
- (c) \( \frac{dy}{dt} = y(2 - y)(3 - y) \)

13.3. Draw a slope field for each of the given differential equations:

- (a) \( \frac{dy}{dt} = 2 + 3y \)
- (b) \( \frac{dy}{dt} = -y(2 - y) \)
- (c) \( \frac{dy}{dt} = 2 - 3y + y^2 \)
- (d) \( \frac{dy}{dt} = -2(3 - y)^2 \)
- (e) \( \frac{dy}{dt} = y^2 - y + 1 \)
- (f) \( \frac{dy}{dt} = y^3 - y \)
- (g) \( \frac{dy}{dt} = \sqrt{y}(y - 2)(y - 3)^2, \ y \geq 0 \)

13.4. **Linear or Nonlinear:** Identify which of the differential equations in Problems 2 and 3 is linear and which nonlinear.

13.5. For each of the differential equations (a) to (g) in Problem 3, plot \( \frac{dy}{dt} \) as a function of \( y \), draw the motion along the \( y \) axis, identify the steady state(s) and indicate if the motions are toward or away from the steady state(s).

13.6. **Direction field:** The direction field shown in the figure below corresponds to which differential equation?

- (A) \( \frac{dy}{dt} = ry(y + 1) \)
- (B) \( \frac{dy}{dt} = r(y - 1)(y + 1) \)
- (C) \( \frac{dy}{dt} = -r(y - 1)(y + 1) \)
- (D) \( \frac{dy}{dt} = ry(y - 1) \)
- (E) \( \frac{dy}{dt} = -ry(y + 1) \)
13.7. **Differential equation:** Given the differential equation and initial condition

\[
\frac{dy}{dt} = y^2(y - a), \ y(0) = 2a
\]

where \(a > 0\) is a constant, the value of the function \(y(t)\) would

(A) Approach \(y = 0\).

(B) Grow larger with time.

(C) Approach \(y = a\).

(D) Stay the same.

(E) None of the above.

13.8. **There's a hole in the bucket:** Water flows into a bucket at constant rate \(I\). There is a hole in the container. Explain the model

\[
\frac{dh}{dt} = I - k\sqrt{h}.
\]

Analyze the behaviour predicted. What would the height be after a long time? Is this result always valid, or is an additional assumption needed? Hint: recall Example 12.3.

13.9. **Cubical crystal:** A crystal grows inside a medium in a cubical shape with side length \(x\) and volume \(V\). The rate of change of the volume is given by

\[
\frac{dV}{dt} = kx^2 (V_0 - V)
\]

where \(k\) and \(V_0\) are positive constants.

(a) Rewrite this as a differential equation for \(\frac{dx}{dt}\).

(b) Suppose that the crystal grows from a very small “seed.” Show that its growth rate continually decreases.
(c) What happens to the size of the crystal after a very long time?

(d) What is its size (that is, what is either \( x \) or \( V \)) when it is growing at half its initial rate?

13.10. **The Law of Mass Action:** The Law of Mass Action in Section 13.1.3 led to the assumption that the rate of a reaction involving two types of molecules (A and B) is proportional to the product of their concentrations, \( k \cdot a \cdot b \). Explain why the sum of the concentrations, \( k \cdot (a + b) \) would not make for a sensible assumption about the rate of the reaction.

13.11. A biochemical reaction in which a substance \( S \) is both produced and consumed is investigated. The concentration \( c(t) \) of \( S \) changes during the reaction, and is seen to follow the differential equation

\[
\frac{dc}{dt} = K_{\text{max}} \frac{c}{k + c} - rc
\]

where \( K_{\text{max}}, k, r \) are positive constants with certain convenient units. The first term is a concentration-dependent production term and the second term represents consumption of the substance.

(a) What is the maximal rate at which the substance is produced? At what concentration is the production rate 50% of this maximal value?

(b) If the production is turned off, the substance will decay. How long would it take for the concentration to drop by 50%?

(c) At what concentration does the production rate just balance the consumption rate?

13.12. **Logistic growth with proportional harvesting:** Consider a fish population of density \( N(t) \) growing at rate \( g(N) \), with harvesting, so that the population satisfies the differential equation

\[
\frac{dN}{dt} = g(N) - h(N).
\]

Now assume that the growth rate is logistic, so \( g(N) = rN \frac{(K-N)}{K} \) where \( r, K > 0 \) are constant. Assume that the rate of harvesting is proportional to the population size, so that

\[
h(N) = qEN
\]

where \( E \), the effort of the fishermen, and \( q \), the catchability of this type of fish, are positive constants. Use qualitative methods discussed in this chapter to analyze the behaviour of this equation. Under what conditions will this lead to a sustainable fishery?

13.13. **Logistic growth with constant number harvesting:** Consider the same fish population as in the previous problem, but this time assume that the rate of harvesting is fixed, regardless of the population size, so that

\[
h(N) = H
\]

where \( H \) is a constant number of fish being caught and removed per unit time. Analyze this revised model and compare it to the previous results.
13.14. **Scaling time in the logistic equation:** Consider the scaled logistic equation (13.3). Recall that \( r \) has units of 1/time, so \( 1/r \) is a quantity with units of time. Now consider scaling the time variable in (13.3) by defining \( t = s/r \). Then \( s \) carries no units (\( s \) is “dimensionless”). Substitute this expression for \( t \) in (13.3) and find the differential equation so obtained (for \( dy/ds \)).

13.15. **Euler’s method applied to logistic growth:** Consider the logistic differential equation

\[
\frac{dy}{dt} = ry(1 - y).
\]

Let \( r = 1 \). Use Euler’s method to find a solution to this differential equation starting with \( y(0) = 0.5 \), and step size \( h = 0.2 \). Find the values of \( y \) up to time \( t = 1.0 \).

13.16. **Spread of infection:** In the model for the spread of a disease, we used the fact that the total population is constant \( (S(t) + I(t) = N=\text{constant}) \) to eliminate \( S(t) \) and analyze a differential equation for \( I(t) \) on its own. Carry out a similar analysis, but eliminate \( I(t) \). Then analyze the differential equation you get for \( S(t) \) to find its steady states and behavior, practicing the qualitative analysis discussed in this chapter.

13.17. **Vaccination strategy:** When an individual is vaccinated, he or she is “removed” from the susceptible population, effectively reducing the size of the population that can participate in the disease transmission. For example, if a fraction \( \phi \) of the population is vaccinated, then only the remaining \( (1 - \phi)N \) individuals can be either susceptible or infected, so \( S(t) + I(t) = (1 - \phi)N \). When smallpox was an endemic disease, it had a basic reproductive number of \( R_0 = 7 \). What fraction of the population would have had to be vaccinated to eradicate this disease?

13.18. **Social media:** Sally Sweetstone has invented a new type of social media App called HeadSpace, which instantly matches compatible mates according to their changing tastes and styles. Users hear about the App from one another by word of mouth and sign up for an account. The account expires randomly, with a half-life of 1 month. Suppose \( y_1(t) \) are the number of individuals who are not subscribers and \( y_2(t) \) are the number of subscribers at time \( t \). The following model has been suggested for the evolving subscriber population

\[
\frac{dy_1}{dt} = by_2 - ay_1y_2, \\
\frac{dy_2}{dt} = ay_1y_2 - by_2.
\]

(a) Explain the terms in the equation. What is the value of the constant \( b \)?

(b) Show that the total population \( P = y_1(t) + y_2(t) \) is constant. (This is a conservation statement.)

(c) Use the conservation statement to eliminate \( y_1 \). Then analyze the differential equation you obtain for \( y_2 \).

(d) Use your model to determine whether this newly launched social media will be successful or whether it will go extinct.
13.19. **A bimolecular reaction**: Two molecules of A can react to form a new chemical, B. The reaction is **reversible** so that B also continually decays back into 2 molecules of A. The differential equation model proposed for this system is

\[
\frac{da}{dt} = -\mu a^2 + 2\beta b \\
\frac{db}{dt} = \mu \frac{a^2}{2} - \beta b,
\]

where \(a(t), b(t) > 0\) are the concentrations of the two chemicals.

(a) Explain the factor 2 that appears in the differential equations and the conservation statement. Show that the total mass \(M = a(t) + 2b(t)\) is constant.

(b) Use the techniques in this chapter to investigate what happens in this chemical reaction, to find any steady states, and to explain the behaviour of the system.