Chapter 10
Exponential functions

“The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth.”


In this chapter, we introduce an important new class of functions, the exponential functions. We first describe the discrete process of population doubling, represented by $2^n$, where $n$ is some integer. We then generalize to a continuous function $2^x$ where $x$ is any real number. Once a continuous function is defined, we can attach a meaning to the idea of its derivative. Computing the derivative of an exponential function, we encounter a specially convenient base denoted $e$, leading us to adopt the exponential function $e^x$. We discuss applications of these ideas to unlimited growth of bacteria and other populations.

10.1 Unlimited growth and doubling

<table>
<thead>
<tr>
<th>Section 10.1 Learning goals</th>
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<tbody>
<tr>
<td>1. Understand the link between population doubling and integer powers of base 2.</td>
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<tr>
<td>2. Given information about the doubling time of a population and its initial size, be able to determine the size of that population at some later generation.</td>
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<tr>
<td>3. Appreciate the connection between $2^n$ for integer values of $n$ and $2^x$ for a real number $x$.</td>
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10.1.1 The Andromeda Strain

The Andromeda Strain scenario (described by Crichton in the opening quotation) motivates our investigation of population doubling and uncontrolled growth. Consider $2^n$ where $n = 1, 2, \ldots$ is an integer. Note that the “variable” $n$ appears in the exponent. We list the first ten values of this discrete function in Table 10.1. It is clear that an initially “gentle” growth becomes extremely steep in just a few steps, as shown in the accompanying graph.

![Graph of $y=2^n$](image)

<table>
<thead>
<tr>
<th>$n$</th>
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<tbody>
<tr>
<td>0</td>
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<td>9</td>
<td>512</td>
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<td>10</td>
<td>1024</td>
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Table 10.1. Powers of 2 including both negative and positive integers: Here we show $2^n$ for $-4 < n < 10$. Note that $2^{10} \approx 1000 = 10^3$. This is a useful approximation in converting binary numbers (powers of 2) to decimal numbers (powers of 10).

Aside from the rapid growth of the function $2^n$, we also notice that $2^{10} \approx 1000 = 10^3$, which helps with simple approximations. With this preparation, we can now check the accuracy of the Crichton statement about bacterial growth.

Example 10.1 (Growth of E. coli:) Use the following facts to check the assertion made by Michael Crichton in the quotation at the beginning of this chapter.

- Mass of 1 E. coli cell: 1 nanogram = $10^{-9}$ gm = $10^{-12}$ kg.
- Mass of Planet Earth: $6 \cdot 10^{24}$ kg.
Solution: Based on the above two facts, we surmise that the size of an E. coli colony (number of cells) that together form a mass equal to Planet Earth would be

\[
m = 6 \cdot 10^{24} \text{ kg} \cdot \frac{10^{-12}}{\text{kg}} = 6 \cdot 10^{36}.
\]

Each hour corresponds to 3 twenty-minute generations. In a period of 24 hours, there are \(24 \times 3 = 72\) generations, each doubling the colony size. After 1 day of uncontrolled growth, the number of cells would be \(2^{72}\). We can find a decimal approximation using the observation that \(2^{10} \approx 10^3\):

\[
2^{72} = 2^2 \cdot 2^{70} = 4 \cdot (2^{10})^7 \approx 4 \cdot (10^3)^7 = 4 \cdot 10^{21}.
\]

Using a scientific calculator, the value is found to be \(4.7 \cdot 10^{21}\), so the approximation is relatively good.

Apparently, the estimate made by Crichton is not quite accurate. However, it can be shown that it takes less than 2 days to produce a number far in excess of the desired size. (The exact number of generations is left as an exercise for the reader and is discussed later in Example 10.12.)

10.1.2 The function \(2^x\) and its “relatives”

We would like to generalize the function \(2^x\) to a continuous function, so that tools of calculus such as derivatives can be used. To do so, we start with values that can be calculated

![Graphs of discrete and smooth functions](image)

Figure 10.1. (a) Values of the function \(2^x\) for discrete value of \(x\). We can compute many values (e.g. for \(x = 0, \pm 1, \pm 2\), by simple arithmetical operations, and for \(x = \pm 1/2, \pm 3/2\) by evaluating square roots). (b) The function \(2^x\) is connected smoothly to form a continuous curve.
based on previous mathematical experience, and then “fill in gaps”. From previous familiarity with power functions such as \( y = x^2 \) (not to be confused with \( 2^x \)), we know the value of

\[
2^{1/2} = \sqrt{2} \approx 1.41421 \ldots
\]

We can use this value to compute

\[
2^{3/2} = (\sqrt{2})^3, \quad 2^{5/2} = (\sqrt{2})^5,
\]

and all other fractional exponents that are multiples of 1/2. We can add these to the graph of our previous powers of 2 to fill in additional points. This is shown on Figure 10.1(a).

In this way, we could also calculate exponents that are multiples of 1/4 since

\[
2^{1/4} = \sqrt[4]{2}
\]

is a value that we can obtain. We show how adding these values leads to an even finer set of points. By continuing in the same way, we fill in the graph of the emerging function. Connecting the dots smoothly allows us to define a value for any real \( x \), of a new continuous function,

\[
y = f(x) = 2^x.
\]

here \( x \) is no longer restricted to an integer. This function is shown in Figure 10.1(b) as the smooth curve superimposed on the points we have gathered.

**Example 10.2 (Generalization to other bases)** Use similar ideas to plot “relatives” of \( 2^x \) that have other bases, such as \( y = 3^x, y = 4^x \) and \( y = 10^x \) and comment about the function \( y = a^x \) where \( a > 0 \) is a constant (called the base).

**Solution:** We first form the discrete function \( a^n \) for integer values of \( n \), simply by multiplying \( a \) by itself \( n \) times. This is analogous to Fig. 10.1. So long as \( a \) is positive, we can fill in values of \( a^x \) when \( x \) is rational (in the same way as we did for \( 2^x \)), and we can smoothly connect the points to lead to the continuous function \( a^x \) for any real \( x \). Given some positive constant \( a \), we define the new function \( f(x) = a^x \) as the exponential function with base \( a \). Shown in Figure 10.2 are the functions \( y = 2^x, y = 3^x, y = 4^x \) and \( y = 10^x \).
10.2 Derivatives of exponential functions and the function $e^x$

Section 10.2 Learning goals

1. Using the definition of the derivative, calculate the derivative of the function $y = a^x$ for an arbitrary base $a > 0$.

2. Understand the significance of the special base $e$.

3. Learn the properties of the function $e^x$, its derivatives, and how to manipulate it algebraically.

4. Note the fact that the function $y = e^{kx}$ has a derivative that is proportional to the same function.

10.2.1 Calculating the derivative of $a^x$

In this section we show how to compute the derivative of the new exponential function. Rather than restricting attention to the special case $y = 2^x$, we consider an arbitrary positive constant $a$ as the base.\footnote{The base has to be positive, to ensure that the function is defined for all real $x$.} For $a > 0$ let

$$y = f(x) = a^x.$$
Then

\[
\frac{da^x}{dx} = \lim_{h \to 0} \frac{(a^{x+h} - a^x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(a^x a^h - a^x)}{h}
\]

\[
= \lim_{h \to 0} a^x \frac{(a^h - 1)}{h}
\]

\[
= a^x \left[ \lim_{h \to 0} \frac{a^h - 1}{h} \right].
\]

Notice that the variable \(x\) appears only in the common factor \(a^x\) which we pulled out of the expression. The limit applies to \(h\), not \(x\), allowing us to do so. Everything inside the square brackets depends only on the base we used. Once the limit is evaluated, that term in square brackets is some constant that we will denote \(C_a\). To summarize, we have found that

The derivative of an exponential function \(a^x\) is of the form \(C_a a^x\) where \(C_a\) is a constant that depends only on the base \(a\).

We now examine this in more detail with the bases 2 and 10.

**Example 10.3 (Derivative of \(2^x\))** Compute the derivative for the base \(a = 2\) using the above result.

**Solution:** For base \(a = 2\), we have

\[
\frac{d2^x}{dx} = C_2 \cdot 2^x
\]

where

\[
C_2(h) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx \frac{2^h - 1}{h} \text{ for small } h.
\]

We still need to find the decimal expansion of \(C_2\). We do so in the next example.

**Example 10.4 (The value of \(C_2\))** Find an approximation for the value of the constant \(C_2\) in Example 10.3 by calculating the value of the ratio \((2^h - 1)/h\) for small (finite) values of \(h\), e.g., \(h = 0.1, 0.01, \text{ etc.}\) Do these successive approximations for \(C_2\) value approach a fixed real number?

**Solution:** We take these successively smaller values of \(h\) and compute \((2^h - 1)/h\) on a scientific calculator. This approximates the constant \(C_2\) with increasing levels of accuracy. We find that \(h = 0.1\) leads to \(C_2 \approx 0.7177\), \(h = 0.001\) leads to \(C_2 \approx 0.6934\), and \(h = 0.00001\) to \(C_2 \approx 0.6931\). We see that these approximations approach a fixed value, \(C_2 \approx 0.6931\). (The actual value has an infinitely long decimal expansion that we here represent by its first few digits.) Thus, the derivative of \(2^x\) is

\[
\frac{d2^x}{dx} = C_2 2^x \approx (0.6931) \cdot 2^x.
\]
10.2. Derivatives of exponential functions and the function $e^x$

Example 10.5 (The base 10 and the derivative of $10^x$) Determine the derivative of $y = f(x) = 10^x$.

Solution: For base 10 we have

$$C_{10}(h) \approx \frac{10^h - 1}{h}$$

for small $h$.

We find, by similar approximation, that $C_{10} = 2.3026$, so that

$$\frac{d10^x}{dx} = C_{10} \cdot 10^x = (2.3026) \cdot 10^x.$$

Thus, the derivative of $y = a^x$ is proportional to the same function, but the constant of proportionality ($C_a$) depends on the base.

10.2.2 The natural base $e$ is convenient for calculus

In Examples 10.3-10.5, we found that the derivative of $a^x$ is $C_a a^x$, where the constant $C_a$ depends on the base. These constants are somewhat inconvenient, but unavoidable if we use an arbitrary base. Here we ask whether there exists a convenient base (to be called “$e$”) for which the constant is particularly simple, namely, such that $C_e = 1$. This indeed, is the property of the natural base that we identify as follows.

Such a base would have to have the property that

$$C_e = \lim_{h \to 0} \frac{e^h - 1}{h} = 1,$$

so for small $h$

$$\frac{e^h - 1}{h} \approx 1.$$

This means that

$$e^h - 1 \approx h \implies e^h \approx h + 1 \implies e \approx (1 + h)^{1/h}.$$

More formally,

$$e = \lim_{h \to 0} (1 + h)^{1/h}. \quad (10.1)$$

We can find an approximate decimal expansion for $e$ by calculating the ratio in (10.1) for some very small (but finite value) of $h$ on a scientific calculator. We find (e.g. for $h = 0.00001$) that

$$e \approx (1.00001)^{100000} \approx 2.71826.$$

To summarize, we have found that for the special base, $e$, we have the following property:

The derivative of the function $e^x$ is $e^x$.

The value of base $e$ is obtained from the limit in Eqn. (10.1). It is straightforward to show that this can be written in either of two equivalent forms

The base of the natural exponential function is the number defined as follows:

$$e = \lim_{h \to 0} (1 + h)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$
10.2.3 Properties of the function $e^x$

We list below some of the key features of the function $y = e^x$: All of these stem from basic manipulations of exponents as reviewed in Appendix B.A.

1. $e^a e^b = e^{a+b}$ as with all similar exponent manipulations.

2. $(e^a)^b = e^{ab}$ also stems from simple rules for manipulating exponents.

3. $e^x$ is a function that is defined, continuous, and differentiable for all real numbers $x$.

4. $e^x > 0$ for all values of $x$.

5. $e^0 = 1$, and $e^1 = e$.

6. $e^x \to 0$ for increasing negative values of $x$.

7. $e^x \to \infty$ for increasing positive values of $x$.

8. The derivative of $e^x$ is $e^x$. (Shown in this chapter).

Example 10.6 Find the derivative of $e^x$ at $x = 0$ and show that the tangent line at that point is the line $y = x + 1$.

Solution: The derivative of $e^x$ is $e^x$, and at $x = 0$ the slope of the tangent line is therefore $e^0 = 1$. The tangent line goes through $(0, e^0) = (0, 1)$ so it has a $y$ intercept of 1. Thus the tangent line at $x = 0$ with slope 1 is $y = x + 1$. This is shown in Figure 10.3.

Figure 10.3. The function $y = e^x$ has the property that its tangent line at $x = 0$ has slope 1. (Note that the horizontal scale on this graph is $-4 \leq x \leq 4$.)
10.2. Derivatives of exponential functions and the function $e^x$

10.2.4 Composite derivatives involving exponentials

Using the derivative of $e^x$ and the chain rule, we can now differentiate composite functions in which the exponential appears.

Example 10.7 Find the derivative of $y = e^{kx}$. □

Solution: The simple chain rule with $u = kx$ leads to

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

but

$$\frac{du}{dx} = k \quad \text{so} \quad \frac{dy}{dx} = e^u k = ke^{kx}.$$ 

This is a useful result, which we highlight for future use:

The derivative of $y = e^{kx}$ is $dy/dx = ke^{kx}$.

Example 10.8 (Chemical reactions) According to the collision theory of bimolecular gas reactions, a reaction between two molecules occurs when the molecules collide with energy greater than some activation energy, $E_a$, referred to as the Arrhenius activation energy. $E_a > 0$ is constant for the given substance. The fraction of bimolecular reactions in which this collision energy is achieved is

$$F = e^{-\left(\frac{E_a}{RT}\right)}$$

where $T$ is temperature (in degrees Kelvin) and $R > 0$ is the gas constant. Suppose that the temperature $T$ increases at some constant rate $C$ per unit time. Determine the rate of change of the fraction $F$ of collisions that result in a successful reaction. □

Solution: This is a related rates problem involving an exponential function that depends on the temperature, which depends on time, $F = e^{-\left(\frac{E_a}{RT(t)}\right)}$. We are asked to find the derivative of $F$ with respect to time when the temperature increases. We are given that $dT/dt = C$. Let $u = -E_a/RT$. Then $F = e^u$. Using the chain rule,

$$\frac{dF}{dt} = \frac{dF}{du} \frac{du}{dT} \frac{dT}{dt}.$$ 

Further, we have $E_a, R, C$ constants, so

$$\frac{dF}{du} = e^u \quad \text{and} \quad \frac{du}{dT} = \frac{E_a}{RT^2}.$$ 

Assembling these parts, we have

$$\frac{dF}{dt} = e^u \frac{E_a}{RT^2} C = C \frac{E_a}{R} T^{-2} e^{-\left(\frac{E_a}{RT}\right)} = \frac{CE_a}{RT^2} e^{-\left(\frac{E_a}{RT}\right)}.$$
10.2.5 The function $e^x$ satisfies a new kind of equation

We divert our attention to an interesting observation before continuing the development of this chapter. We have seen that the function

$$y = f(x) = e^x$$

satisfies the relationship

$$\frac{dy}{dx} = f'(x) = f(x) = y.$$ 

In other words, when differentiating, we get the same function back again. Summarizing,

<table>
<thead>
<tr>
<th>The function $y = f(x) = e^x$ is equal to its own derivative and hence, it satisfies the equation $\frac{dy}{dx} = y$.</th>
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</thead>
<tbody>
<tr>
<td>An equation linking a function and its derivative(s) is called a <strong>differential equation</strong>.</td>
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</tbody>
</table>

This is a new type of equation, unlike ones seen before in this course. We will see shortly (In Chapters 11-13) that such equations have important applications in biology, physics, chemistry, and science in general.

10.3 Inverse functions and logarithms

So far in this chapter, we have defined the new function $e^x$ and computed its derivative. Paired with this newcomer is an inverse function, the natural logarithm, $\ln(x)$. The reader will find it helpful to review concepts of inverse functions in Appendix C. In particular, the following key ideas are important:

Given a function $y = f(x)$, its inverse function, denoted $f^{-1}(x)$ satisfies

$$f(f^{-1}(x)) = x, \quad \text{and} \quad f^{-1}(f(x)) = x.$$ 

The range of $f(x)$ is the domain of $f^{-1}(x)$ (and vice versa), which implies that in many cases, the relationship holds only on some subset of real numbers. An discussed in the appendix, the domain of a function (such as $y = x^2$) has to be restricted (e.g. to $x \geq 0$) so that its inverse function ($y = \sqrt{x}$) be defined. On that restricted domain, the graphs of $f$ and $f^{-1}$ are mirror images of one another about the line $y = x$. Essentially, this stems from the fact that the roles of $x$ and $y$ are interchanged.
Section 10.3 Learning goals

1. Understand the concept of inverse function from both algebraic and geometric points of view: given a function, be able to determine whether (and for what restricted domain) an inverse function can be defined and to sketch its inverse function. (Review Appendix C.E).

2. Understand the relationship between the domain and range of a function and the range and domain of its inverse function. (Review Appendix C.E).

3. Be able to apply these ideas to the logarithm, which is the inverse of an exponential function.

4. Follow and be able to reproduce the calculation of the derivative of $\ln(x)$ using implicit differentiation.

10.3.1 The natural logarithm is an inverse function for $e^x$

For our newly defined function $y = f(x) = e^x$ we will define an inverse function, shown on Figure 10.4. We will call this function the logarithm (base $e$), and write it as

$$y = f^{-1}(x) = \ln(x).$$

We have the following connection: $y = e^x$ implies $x = \ln(y)$. The fact that the functions $y = e^x$ and $y = \ln(x)$ are inverses of each other is illustrated in Figure 10.4.

Figure 10.4. The function $y = e^x$ is shown here together with its inverse, $y = \ln x$. 

are inverses also implies that

\[ e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x. \]

The domain of \( e^x \) is \( -\infty < x < \infty \), and its range is \( x > 0 \). For the inverse function, this domain and range are interchanged, meaning that \( \ln(x) \) is only defined for \( x > 0 \) (its domain) and returns values in \( -\infty < x < \infty \) (its range). As shown in Fig. 10.4, the functions \( e^x \) and \( \ln(x) \) are reflections of one another about the line \( y = x \).

Properties of the logarithm stem directly from properties of the exponential function. A brief review of these is provided in Appendix B.B. Briefly,

1. \( \ln(ab) = \ln(a) + \ln(b) \),
2. \( \ln(a^b) = b \ln(a) \),
3. \( \ln(1/a) = \ln(a^{-1}) = -\ln(a) \).

### 10.3.2 Derivative of \( \ln(x) \) by implicit differentiation

Implicit differentiation is helpful whenever an inverse function appears. Knowing the derivative of the original function allows us to easily compute the derivative of its inverse by using the special relationship. Here we use implicit differentiation to find the derivative of \( y = \ln(x) \). First, restate the relationship in the inverse form, but consider \( y \) as the dependent variable, that is think of \( y \) as a quantity that depends on \( x \):

\[ y = \ln(x) \implies e^y = x \implies \frac{d}{dx} e^{y(x)} = \frac{d}{dx} x. \]

Applying the chain rule,

\[ \frac{d e^y}{d y} \frac{dy}{dx} = 1 \implies e^y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \]

We have thus shown the following:

| The derivative of \( \ln(x) \) is \( 1/x \): |
| \[ \frac{d \ln(x)}{dx} = \frac{1}{x} \] |
10.4 Applications of the logarithm

Section 10.4 Learning goals

1. Understand the relationships between properties of $e^x$ and properties of its inverse $\ln(x)$, and master manipulations of expressions involving both.

2. Be able to use logarithms for base conversions, and for solving equations involving the exponential function. (For instance, given an equation of the form $A = e^{bt}$, solve for $t$.)

3. Given a relationship such as $y = ax^b$, show that $\ln(y)$ is related linearly to $\ln(x)$, and use data points for $(x, y)$ to determine the values of $a$ and $b$.

10.4.1 Using the logarithm for base conversion

The logarithm is helpful in changing an exponential function from one base to another. We give some examples here.

Example 10.9 Rewrite $y = 2^x$ in terms of base $e$.  

**Solution:** We apply $\ln$ and then exponentiate the result. Manipulations of exponents and logarithms lead to the desired results as follows:

$$y = 2^x \Rightarrow \ln(y) = \ln(2^x) = x \ln(2).$$

$$e^{\ln(y)} = e^{x \ln(2)} \Rightarrow y = e^{x \ln(2)}.$$  

We find (using a calculator) that $\ln(2) = 0.6931$. This coincides with the value we computed earlier for $C_2$ in Example 10.4, so we have

$$y = e^{kx} \quad \text{where} \quad k = \ln(2) = 0.6931.$$  

Example 10.10 Find the derivative of $y = 2^x$.  

**Solution:** We have expressed this function in the alternate form

$$y = 2^x = e^{kx} \quad \text{with} \quad k = \ln(2).$$

From Example 10.7 we have

$$\frac{dy}{dx} = k e^{kx} = \ln(2)e^{\ln(2)x} = \ln(2)2^x.$$  

Through the above base conversion and chain rule, we were able to relate the constant $C_2$ in Example 10.4 to the natural logarithm of 2: $C_2 = \ln(2)$. 
10.4.2 The logarithm helps to solve exponential equations

Equations involving the exponential function can sometimes be simplified and solved using the logarithm. Here we provide a few examples of this kind.

**Example 10.11** Find zeros of the function \( y = f(x) = e^{2x} - e^{5x^2} \).

**Solution:** We are being asked to find values of \( x \) for which \( f(x) = e^{2x} - e^{5x^2} = 0 \). We write

\[
e^{2x} - e^{5x^2} = 0 \quad \Rightarrow \quad e^{2x} = e^{5x^2} \quad \Rightarrow \quad \frac{e^{5x^2}}{e^{2x}} = 1 \quad \Rightarrow \quad e^{5x^2-2x} = 1
\]

Taking logarithm of both sides, and using the facts that \( \ln(e^{5x^2-2x}) = 5x^2 - 2x \) and \( \ln(1) = 0 \), we obtain

\[
e^{5x^2-2x} = 1 \quad \Rightarrow \quad 5x^2 - 2x = 0 \quad \Rightarrow \quad x = 0, 5/2.
\]

We see that the logarithm is useful in the last step of isolating \( x \), after simplifying the exponential expressions appearing in the equation.

**Andromeda Strain, revisited**

Earlier in this chapter we had posed the question: How long will it take for the Andromeda strain population to attain a size of \( 6 \cdot 10^{36} \) cells, i.e. to grow to an Earth-sized colony. We now solve this problem using the continuous exponential function and the logarithm.

Recall that the bacterial doubling time is 20 min. If time is measured in minutes, the number, \( B(t) \) of bacteria at time \( t \) could be described by the smooth function:

\[
B(t) = 2^{t/20}.
\]

(Note that at \( t = 0 \), we have \( B(t) = 2^0 = 1 \) cell. Further, this function agrees with our previous table and graph for powers of 2 at \( t = 20, 40, 60, 80 \ldots \) minutes, that is, for all integer multiples of the doubling time.)

**Example 10.12 (The Andromeda strain)** Starting from a single cell, how long will it take for an E. coli colony to reach size of \( 6 \cdot 10^{36} \) cells by doubling every 20 minutes?

**Solution:** We can compute the time it takes by solving for \( t \) in \( B(t) = 6 \cdot 10^{36} \), as shown below.

\[
6 \cdot 10^{36} = 2^{t/20} \quad \Rightarrow \quad \ln(6 \cdot 10^{36}) = \ln(2^{t/20})
\]

\[
\ln(6) + 36 \ln(10) = \frac{t}{20} \ln(2),
\]

so

\[
t = 20 \frac{\ln(6) + 36 \ln(10)}{\ln(2)} = 20 \frac{1.79 + 36(2.3)}{0.693} = 2441.27 \min = \frac{2441.27}{60} \text{ hr}.
\]

Hence, it takes nearly 41 hours (but less than 2 days) for the colony to “grow to the size of planet Earth” (assuming the implausible scenario of unlimited growth).
10.4. Applications of the logarithm

Example 10.13 (Using base $e$): Express the number of bacteria in terms of base $e$ (for practice with base conversions).

Solution: We would do this as follows:

$$B(t) = 2^{t/20} \Rightarrow \ln(B(t)) = \frac{t}{20} \ln(2),$$

$$e^{\ln(B(t))} = e^{\frac{t}{20} \ln(2)} \Rightarrow B(t) = e^{kt} \text{ where } k = \frac{\ln(2)}{20} \text{ per min.}$$

The constant $k$ has units of 1/time. We will referred to $k$ as the growth rate of the bacteria. We observe that this constant can be written as:

$$k = \frac{\ln(2)}{\text{doubling time}}.$$  

We will see the usefulness of this approach very soon.

10.4.3 Logarithms help plot data that varies on large scale

Living organisms come in a variety of sizes, from the tiniest cells to the largest whales. Comparing attributes across species of vastly different sizes poses a challenge, as visualizing such data on a simple graph obscures both extremes. Suppose we wish to compare the physiology of organisms of various sizes, from that of a mouse to that of an elephant. An example of such data for metabolic rate versus mass of the animal is shown in Table 10.2.

It would be hard to see all data points clearly on a regular graph. For this reason, it can be helpful to use logarithmic scaling for either or both variables. We show an example of this kind of log-log plot in Figure 10.5.

<table>
<thead>
<tr>
<th>animal</th>
<th>body weight $M$ (gm)</th>
<th>basal metabolic rate (BMR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mouse</td>
<td>25</td>
<td>1580</td>
</tr>
<tr>
<td>rat</td>
<td>226</td>
<td>873</td>
</tr>
<tr>
<td>rabbit</td>
<td>2200</td>
<td>466</td>
</tr>
<tr>
<td>dog</td>
<td>11700</td>
<td>318</td>
</tr>
<tr>
<td>man</td>
<td>70000</td>
<td>202</td>
</tr>
<tr>
<td>horse</td>
<td>700000</td>
<td>106</td>
</tr>
</tbody>
</table>

Table 10.2. Animals of various sizes (mass $M$ in gm) have widely different basal metabolic rates (BMR, generally measured in terms of oxygen consumption rate, i.e. ml $O_2$ consumed per hr). A log-log plot of this data is shown in Fig. 10.5.

In allometry, it is conjectured that such data fits some power function of the form

$$y \approx ax^b, \text{ where } a, b > 0. \quad (10.2)$$

(Note that this is not an exponential function, but a power function with power $b$ and coefficient $a$.) Finding the allometric constants $a$ and $b$ for such a relationship is sometimes useful. Below we illustrate how this can be done based on the graph in Fig 10.5.
Figure 10.5. A log-log plot of the data in Table 10.2, showing ln(BMR) versus ln(M).

Example 10.14 (Log transformation) Define $Y = \ln(y)$, $X = \ln(x)$. Show that Eqn. (10.2) can be rewritten as a linear relationship between $Y$ and $X$.

Solution: We have

$$Y = \ln(y) = \ln(ax^b) = \ln(a) + \ln(x^b) = \ln(a) + b \ln(x) = A + bX$$

where $A = \ln(a)$. Thus, we have shown that $X$ and $Y$ are related linearly:

$$Y = A + bX,$$

where $A = \ln(a)$.

This is the equation of a straight line whose slope is $b$ and whose $Y$ intercept is $A$.

Example 10.15 (Finding the constants) Use the straight line superimposed on the data in Fig. 10.5 to estimate the values of the constants $a$ and $b$.

Solution: We use the straight line that has been fitted to the data in Fig. 10.5. The $Y$ intercept of this line is roughly 8.2. The line goes through the points (20,3) and (0,8.2) (open dots on plot) so its slope is $\approx (3 - 8.2)/20 = -0.26$. According to the relationship we found in Example 10.14,

$$8.2 = A = \ln(a) \Rightarrow a = e^{8.2} = 3640, \quad \text{and} \quad b = -0.26.$$ 

Thus, reverting to the original allometric relationship leads to

$$y = ax^b = 3640x^{-0.26} = \frac{3640}{x^{0.26}}.$$

It is clear that the metabolic rate $y$ decreases with the size $x$ of the animal, as indicated by the data in Table 10.2.
Exercises

10.1. **Polymerase Chain reaction (PCR):** The polymerase chain reaction (PCR) was invented by Mullis in 1983 to amplify DNA. The idea is based on the fact that each strand of (double-stranded) DNA can act as a template for the synthesis of the second (“complementary”) strand. The method consists of repeated cycles of heating (which separates the DNA strands) and cooling (allowing for new DNA to be assembled on each strand). The reaction mixture includes the original DNA to be amplified, plus enzymes and nucleotides, the components needed to form the new DNA. Each cycle doubles the amount of DNA.

A particular PCR experiment consisted of 35 cycles. By what factor was the original DNA amplified? Give your answer both in terms of powers of 2 and approximate decimal (powers of ten) notations. Use the approximation in the caption of Table 10.1 (rather than a scientific calculator) to find the decimal approximation.

10.2. **Invention of the game of chess:** According to some legends, the inventor of the game of chess (who lived in India thousands of years ago) so please his ruler, that he was asked to chose his reward. “I would be content with grains of wheat. Let one grain be placed on the first square of my chess board, and double that number on the second, double that on the third, and so on”, said the inventor. The ruler gladly agreed. A chessboard has $8 \times 8$ squares. How many grains of wheat would be required for the last square on that board? \[ \text{Give your answer in decimal notation.} \]

10.3. In order to produce the graph of the continuous function $2^x$ in Fig. 10.1, it is desirable to generate many points on that graph using simple calculations. (Then those points were connected smoothly). Suppose you have an ordinary calculator with the operations $+,-,\times,\div$. You also know that $\sqrt{2} \approx 1.414$. How would you compute $2^x$ for the values $x = 7/2, x = -1/2, x = -5$?

10.4. Explain the requirement that $a$ must be positive in the exponential function $y = a^x$. What could go wrong if $a$ was a negative base?

10.5. **Derivative of $3^x$:** Find the derivative of $y = 3^x$. What is the value of the multiplicative constant $C_3$ that shows up in your calculation?

10.6. Graph the following functions:

(a) $f(x) = x^2 e^{-x}$

(b) $f(x) = \ln(e^{2x})$

10.7. Express the following in terms of base $e$:

(a) $y = 3^x$

(b) $y = \frac{1}{7^x}$

(c) $y = 15x^2 + 2$

Express the following in terms of base 2:

(d) $y = 9^x$

---

\[38\text{In the original wheat and chessboard problem, we are asked to find the total number of wheat grains on all squares. This requires summing a geometric series, and is a problem ideal for early 2nd term calculus.}\]
(e) \( y = 8^x \)
(f) \( y = -e^{x^2+3} \)

Express the following in terms of base 10:
(g) \( y = 21^x \)
(h) \( y = 1000^{-10x} \)
(i) \( y = 50^{x^2-1} \)

10.8. Compare the values of each pair of numbers (i.e. indicate which is larger):
(a) \( 5^{0.75}, 5^{0.65} \)
(b) \( 0.4^{-0.2}, 0.4^{0.2} \)
(c) \( 1.001^{2}, 1.001^{3} \)
(d) \( 0.999^{1.5}, 0.999^{2.3} \)

10.9. Rewrite each of the following equations in logarithmic form:
(a) \( 3^4 = 81 \)
(b) \( 3^{-2} = \frac{1}{9} \)
(c) \( 27^{-\frac{1}{3}} = \frac{1}{3} \)

10.10. Solve the following equations for \( x \):
(a) \( \ln x = 2 \ln a + 3 \ln b \)
(b) \( \log_a x = \log_a b - \frac{2}{3} \log_a c \)

10.11. Reflections and transformations: What is the relationship between the graph of \( y = 3^x \) and the graph of each of the following functions?
(a) \( y = -3^x \)
(b) \( y = 3^{-x} \)
(c) \( y = 3^{1-x} \)
(d) \( y = 3^{\lfloor x \rfloor} \)
(e) \( y = 2 \cdot 3^x \)
(f) \( y = \log_3 x \)

10.12. Solve the following equations for \( x \):
(a) \( e^{3-2x} = 5 \)
(b) \( \ln(3x - 1) = 4 \)
(c) \( \ln(\ln(x)) = 2 \)
(d) \( e^{ax} = Ce^{bx}, \text{ where } a \neq b \text{ and } C > 0. \)

10.13. Find the first derivative for each of the following functions:
(a) \( y = \ln(2x + 3)^3 \)
(b) \( y = \ln^3(2x + 3) \)
(c) \( y = \ln(\cos \frac{1}{2}x) \)
(d) \( y = \log_a(x^3 - 2x) \) (Hint: \( \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \))
(e) \( y = e^{3x^2} \)
(f) \( y = a^{-\frac{1}{2}x} \)
Exercises

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(g) \( y = x^3 \cdot 2^x \)
(h) \( y = e^{ex} \)
(i) \( y = \frac{e^t - e^{-t}}{e^t + e^{-t}} \)

10.14. Find the maximum and minimum points as well as all inflection points of the following functions:

(a) \( f(x) = x(x^2 - 4) \)
(b) \( f(x) = x^3 - \ln(x), x > 0 \)
(c) \( f(x) = xe^{-x} \)
(d) \( f(x) = \frac{1}{1-x} + \frac{1}{1+x}, -1 < x < 1 \)
(e) \( f(x) = x - 3\sqrt{x} \)
(f) \( f(x) = e^{-2x} - e^{-x} \)

10.15. Shown in Figure 15 is the graph of \( y = Ce^{kt} \) for some constants \( C, k \), and a tangent line. Use data from the graph to determine \( C \) and \( k \).

10.16. Consider the two functions

(a) \( y_1(t) = 10e^{-0.1t} \),
(b) \( y_2(t) = 10e^{0.1t} \).

Which one is decreasing and which one is increasing? In each case, find the value of the function at \( t = 0 \). Find the time at which the increasing function has doubled from this initial value. Find the time at which the decreasing function has fallen to half of its initial value. [Remark: these values of \( t \) are called, the doubling time, and the half-life, respectively]
10.17. **Invasive species**: An ecosystem with mature trees has a relatively constant population of beetles (species 1) that number around $10^9$. At $t = 0$, a single reproducing invasive beetle (species 2) is introduced accidentally. If this population initially grows at the exponential rate 

$$N_2(t) = e^{rt}, \quad \text{where } r = 0.5 \text{ per month}$$

how long will it take for species 2 to overtake the population of the resident species 1? Assume exponential growth for the entire duration.

10.18. **Human population growth**: It is sometimes said that the population of humans on Earth is growing exponentially. By this is meant that

$$P(t) = Ce^{rt}, \quad \text{where } r > 0.$$ 

In this problem we investigate this claim. To do so, we will consider the human population starting in year 1800 ($t = 0$). Hence, we ask whether the data in Table 2.4 fits the relationship 

$$P(t) = Ce^{r(t-1800)}, \quad \text{where } t \text{ is time in years and } r > 0.$$ 

(a) Show that the above relationship implies that $\ln(P)$ is a linear function of time, and that $r$ is the slope of the linear relationship. (Hint: take the natural logarithm of both sides of the relationship and simplify.)

(b) Use the data from Table 2.4 for the years 1800 to 2020 to investigate whether $P(t)$ fits an exponential relationship. (Hint: plot $\ln(P)$, where $P$ is human population (in billions) against time $t$ in years. We refer to this process as “transforming the data”.)

(c) A spreadsheet can be used to fit a straight line through the transformed data you produced in (b). Find the best fit for the growth rate parameter $r$ using that option. What are the units of $r$? What is the best fit value of $C$?

(d) Based on your plot of $\ln(P)$ versus $t$ and the best fit values of $r$ and $C$, over what time interval was the population growing more slowly than the overall trend, and when was it growing more rapidly than this same overall trend?

(e) Under what circumstances could an exponentially growing population be **sustainable**?

10.19. **A sum of exponentials**:

Researchers that investigated the molecular motor dynein found that the number of motors $N(t)$ remaining attached to their microtubule tracks at time $t$ (in sec) after a pulse of activation was well described by a double exponential of the form 

$$N(t) = C_1e^{-r_1t} + C_2e^{-r_2t}, \quad t \geq 0.$$ 

They found that $r_1 = 0.1, r_2 = 0.01$ per second, and $C_1 = 75, C_2 = 25$ percent.

(a) Plot this relationship for $0 < t < 8$ min. Which of the two exponential terms governs the behaviour over the first minute? Which dominates in the later phase?
(b) Now consider a plot of $\ln(N(t))$ versus $t$. Explain what you see and what the slopes and other aspects of the graph represent.

10.20. **Exponential Peeling:**

<table>
<thead>
<tr>
<th>time</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>0.1000</td>
<td>57.6926</td>
</tr>
<tr>
<td>0.2000</td>
<td>42.5766</td>
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<td>35.8549</td>
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<td>0.5000</td>
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</tr>
<tr>
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<td>4.7430</td>
</tr>
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<td>4.5000</td>
<td>0.7840</td>
</tr>
<tr>
<td>6.0000</td>
<td>0.2032</td>
</tr>
<tr>
<td>8.0000</td>
<td>0.0336</td>
</tr>
</tbody>
</table>

**Table 10.3. Table for Problem 20.**

You are given the data in Table 10.3 and told that it was generated by a double exponential function of the form

$$N(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}, \quad t \geq 0.$$ 

Use the data to determine the values of the constants $r_1, r_2, C_1, C_2$.

10.21. **Shannon Entropy:** In a recent application of information theory to the field of genomics, a function called the Shannon entropy, $H$, was considered. A given gene is represented as a binary device: it can be either “on” or “off” (i.e. being expressed or not). If $x$ is the probability that the gene is “on” and $y$ is the probability that it is “off”, the Shannon entropy function for the gene is defined as

$$H = -x \log(x) - y \log(y)$$

[Remark: the fact that $x$ and $y$ are probabilities, just means that they satisfy $0 < x \leq 1$, and $0 < y \leq 1$.] The gene can only be in one of these two states, so $x + y = 1$. Use these facts to show that the Shannon entropy for the gene is greatest when the two states are equally probable, i.e. for $x = y = 0.5$.

10.22. **A threshold function:** The response of a regulatory gene to inputs that affect it is not simply linear. Often, the following so called “squashing function” or “threshold function” is used to link the input $x$ to the output $y$ of the gene.

$$y = f(x) = \frac{1}{1 + e^{(ax+b)}}$$

where $a, b$ are constants.

(a) Show that $0 < y < 1$. 

(b) For \( b = 0 \) and \( a = 1 \) sketch the shape of this function.
(c) How does the shape of the graph change as \( a \) increases?

10.23. Sketch the graph of the function \( y = e^{-t} \sin \pi t \).

10.24. **The Mexican Hat:** Find the critical points of the function

\[
y = f(x) = 2e^{-x^2} - e^{-x^2/3}
\]

and determine the value of \( f \) at those critical points. Use these results and the fact that for very large \( x \), \( f \to 0 \) to draw a rough sketch of the graph of this function. Comment on why this function might be called “a Mexican Hat”. (Note: The second derivative is not very informative here, and we will not ask you to use it for determining concavity in this example. However, you may wish to calculate it just for practice with the chain rule.)

10.25. **The Ricker Equation:** In studying salmon populations, a model often used is the Ricker equation which relates the size of a fish population this year, \( x \) to the expected size next year \( y \). (Note that these populations do not change continuously, since all the parents die before the eggs are hatched.) The Ricker equation is

\[
y = \alpha xe^{-\beta x}
\]

where \( \alpha, \beta > 0 \).

(a) Find the value of the current population which maximizes the salmon population next year according to this model.
(b) Find the value of the current population which would be exactly maintained in the next generation.
(c) Explain why a very large population is not sustainable.

10.26. **Spacing in a fish school:** Life in a social group has advantages and disadvantages: protection from predators is one advantage. Disadvantages include competition with others for food or resources. Spacing of individuals in a school of fish or a flock of birds is determined by the mutual attraction and repulsion of neighbors from one another: each individual does not want to stray too far from others, nor get too close.

Suppose that when two fish are at distance \( x > 0 \) from one another, they are attracted with “force” \( F_a \) and repelled with “force” \( F_r \) given by:

\[
F_a = Ae^{-x/a}
\]
\[
F_r = Re^{-x/r}
\]

where \( A, R, a, r \) are positive constants. \( A, R \) are related to the magnitudes of the forces, and \( a, r \) to the spatial range of these effects.

(a) Show that at the distance \( x = a \) the first function has fallen to \((1/e)\) times its value at the origin. (Recall \( e \approx 2.7 \).) For what value of \( x \) does the second function fall to \((1/e)\) times its value at the origin? Note that this is the reason why \( a, r \) are called spatial ranges of the forces.
(b) It is generally assumed that \( R > A \) and \( r < a \). Interpret what this mean about the comparative effects of the forces and sketch a graph showing the two functions on the same set of axes.

(c) Find the distance at which the forces exactly balance. This is called the comfortable distance for the two individuals.

(d) If either \( A \) or \( R \) changes so that the ratio \( R/A \) decreases, does the comfortable distance increase or decrease? (Give reason.)

(e) Similarly comment on what happens to the comfortable distance if \( a \) increases or \( r \) decreases.

10.27. **Seed distribution:** The density of seeds at a distance \( x \) from a parent tree is observed to be

\[
D(x) = D_0 e^{-x^2/a^2},
\]

where \( a > 0, D_0 > 0 \) are positive constants. Insects that eat these seeds tend to congregate near the tree so that the fraction of seeds that get eaten is

\[
F(x) = e^{-x^2/b^2}
\]

where \( b > 0 \). (Remark: These functions are called Gaussian or Normal distributions. The parameters \( a, b \) are related to the “width” of these bell-shaped curves.) The number of seeds that survive (i.e. are produced and not eaten by insects) is

\[
S(x) = D(x)(1 - F(x))
\]

Determine the distance \( x \) from the tree at which the greatest number of seeds survive.

10.28. **Euler’s “e”:** In 1748, Euler wrote a classic book on calculus (“Introductio in Analysin Infinitorum”) in which he showed that the function \( e^x \) could be written in an expanded form similar to an (infinitely long) polynomial:

\[
e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots
\]

Use as many terms as necessary to find an approximate value for the number \( e \) and for \( 1/e \) to 5 decimal places. Remark: we will see later that such expansions, called power series, are central to approximations of many functions.