Chapter 1

Power functions as building blocks

“There is no knowledge that is not power.”
Ralph Waldo Emerson, (1803-1882)

Some of the beautiful architectural marvels built by humans from ancient to modern times though very complicated as a whole, are made of simple component parts – bricks, beams and joints. Similarly, some mathematical structures that seem complicated can be decomposed into simpler subunits whose properties are straightforward. Understanding these component parts and how they fit together to form more interesting structures is an important step in appreciating properties of more complex (mathematical) structures. This central idea forms the theme of the first chapter.

The components that we explore here are power functions. We first study these on their own, and compare their shapes. We examine an immediate application of our analysis to the biological problem of cell size. Then we expand our horizon to consider polynomials and rational functions. Using the power functions as basic building blocks, we construct the family of polynomials, and investigate how their features are inherited from the underlying behaviour of power functions. Here, we begin to develop a few important curve-sketching skills that will be useful throughout this calculus course.

1.1 Power functions

<table>
<thead>
<tr>
<th>Learning goals (LG)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Understand the shapes of power functions relative to one another (Figs. 1.1, 1.3).</td>
</tr>
<tr>
<td>2. Understand the idea that power functions with low powers dominate near the origin, and power functions with high powers dominate far away from the origin. (Figs. 1.1, 1.3).</td>
</tr>
<tr>
<td>3. Be able to find points of intersection of two power functions (Example 1.1).</td>
</tr>
</tbody>
</table>
Let us consider the power functions, that is functions of the form

\[ y = f(x) = x^n, \]

where \( n \) is a positive integer. Power functions are among the most elementary and “elegant” functions. They are easy to calculate, very predictable and smooth, and, from the point of view of calculus, very easy to handle.

From Figure 1.1a, we see that the power functions \((y = x^n \text{ for } n = 2, \ldots, 5)\) intersect at \( x = 0 \) and \( x = 1 \). This is true for all integer powers. The same figure also demonstrates another extremely important fact: the greater the power \( n \), the flatter the graph near the origin and the steeper the graph beyond \( x > 1 \). This can be restated in terms of the relative size of the power functions. We say that close to the origin, the functions with lower powers dominate, while far from the origin, the higher powers dominate.

![Figure 1.1.](image)

**Figure 1.1.** (a) Graphs of a few power functions \( y = x^n \). All intersect at \( x = 0, 1 \). As the power \( n \) increases, the graphs become flatter close to the origin and steeper at large \( x \) values (LG 1). Near the origin, power functions with lower powers dominate over (have a larger value compared to) power functions with higher powers. Far from the origin, power functions with higher powers dominate (LG 2). (b) Graphs of the two power functions \((y = 5x^2, y = 2x^3)\). Close to the origin, the quadratic power function has a larger value, whereas for large \( x \), the cubic function has larger values. The functions intersect when \( 5x^2 = 2x^3 \), which holds for either \( x = 0 \) or \( x = 5/2 = 2.5 \) (LG 3).

More generally, a power function has the form

\[ y = f(x) = K \cdot x^n \]

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1We only need to use multiplication to compute the value of these functions at any point.
1.2. How big can a cell be? A model for nutrient balance

where \( n \) is a positive integer and \( K \), sometimes called the **coefficient** is a constant. So far, we have compared power functions whose coefficient is \( K = 1 \). But we can extend our discussion to a more general case as well.

**Example 1.1** Find points of intersection and compare the sizes of the two power functions

\[
y_1 = ax^n, \quad \text{and} \quad y_2 = bx^m.
\]

where \( a \) and \( b \) are constants. You may assume that both \( a \) and \( b \) are positive. □

**Solution:** This comparison is a slight generalization of what we have seen above. First, we note that the coefficients \( a \) and \( b \) merely scale the vertical behaviour (i.e. stretch the graph along the \( y \) axis. It is still true that the two functions intersect at \( x = 0 \); further, as before, the higher the power, the flatter the graph close to \( x = 0 \), and the steeper for large positive or negative values of \( x \). However, now another point of intersection of the graphs occur when

\[
ax^n = bx^m \quad \Rightarrow \quad x^{n-m} = \frac{b}{a}
\]

We can solve this further to obtain a solution in the first quadrant\(^2\),

\[
x = \left(\frac{b}{a}\right)^{1/(n-m)}.
\]

(1.1)

This is shown in Figure 1.1b for the specific example of \( y_1 = 5x^2 \), \( y_2 = 2x^3 \). If \( b/a \) is a positive then in general the value given by the expression (1.1) is a real number.

**Example 1.2** Determine points of intersection for the following pairs of functions: (a) \( y_1 = 3x^4 \) and \( y_2 = 27x^2 \), (b) \( y_1 = (4/3)\pi x^3 \), \( y_2 = 4\pi x^2 \). □

**Solution:** (a) Intersections occur at \( x = 0 \) and at \( \pm(27/3)^{1/(4-2)} = \pm\sqrt{9} = \pm3 \). (b) These functions intersect at \( x = 0, 3 \) but there are no other intersections at negative values of \( x \).

In many cases, the points of intersection will be irrational numbers whose decimal approximations can only be obtained by a scientific calculator or by some approximation method (such as **Newton’s Method**\(^3\)).

The observations we have made so far already allow us to examine a biological problem related to the size of cells. We see that application of these ideas will provide insight into why cells have a size limitation, as discussed in the next section.

### 1.2 How big can a cell be? A model for nutrient balance

The shapes of living cells are designed to be uniquely suited to their functions. Few cells are really spherical. Many have long appendages, cylindrical parts, or branch-like structures.

\(^2\)As we will shortly see, if \( n, m \) are both even or both odd, there will also be an intersection in the third quadrant, at \( x = -(b/a)^{1/(n-m)} \)

\(^3\)This method will be discussed in Section 5.4.1
But here, we will neglect all these beautiful complexities and look at a simple spherical cell. The question we want to explore is what physical or biological constraints determine the size of a cell and why some size limitations exist. Why should animals be made of millions of tiny cells, instead of just a few hundred large ones?

**Learning goals**

1. Follow and understand the derivation of a mathematical model for cell nutrient absorption and consumption (Section 1.2.1).
2. Develop the skill of using parameters \((k_1, k_2)\) rather than specific numbers in mathematical expressions.
3. Understand the link between power functions in Section 1.1 and cell nutrient balance in the model (Eqs. 1.3).
4. Be able to verbally interpret the results of the model (Section 1.2.2).

![Figure 1.2](image)

**Figure 1.2.** A cell (assumed spherical) absorbs nutrients at a rate proportional to its surface area \(S\), but consumes nutrients at a rate proportional to its volume \(V\). \(k_1, k_2\) are proportionality constants. The surface area and volume of a sphere of radius \(r\) are given by \(S = 4\pi r^2\), \(V = \frac{4}{3}\pi r^3\). These facts are used to assemble a simple model for nutrient balance in a spherical cell.

Here we use a relatively simple mathematical argument to get some insight. To do so, we will formulate a **mathematical model**, a simplified representation of a real situation. The simplification aims to represent the important aspects of the process, while neglecting or idealizing complicating details. Below we follow a reasonable set of assumptions and mathematical facts to explore how nutrient balance can affect and limit cell size.

### 1.2.1 Building the model

In order to build the model we make some simplifying assumptions and then restate them mathematically. We base the model on the following **assumptions**:
1.2. How big can a cell be? A model for nutrient balance

1. The cell is roughly spherical (See Figure 1.2).
2. The cell absorbs oxygen and nutrients through its surface. The larger the surface area, \( S \), the faster the total rate of absorption. We will assume that the rate at which nutrients (or oxygen) are absorbed is proportional\(^4\) to the surface area of the cell.
3. The rate at which nutrients are consumed (i.e., used up) in metabolism is proportional to the volume, \( V \), of the cell. The bigger the volume, the more nutrients are needed to keep the cell alive.

We define the following quantities for our model of a single cell:

\[
A = \text{net rate of absorption of nutrients per unit time,} \\
C = \text{net rate of consumption of nutrients per unit time,} \\
V = \text{cell volume,} \\
S = \text{cell surface area,} \\
r = \text{radius of the cell.}
\]

We now rephrase the assumptions mathematically. By assumption (2), the absorption rate, \( A \), is proportional to \( S \): This means that

\[A = k_1 S,\]

where \( k_1 \) is a constant of proportionality. Since absorption and surface area are positive quantities, in this case only positive values of the proportionality constant make sense, so \( k_1 \) must be positive. (The value of this constant would depend on properties of the cell membrane such as its permeability or how many pores it contains to permit passage of nutrients.) By using a generic parameter to represent this proportionality constant, we keep the model general enough to apply to many different cell types. (LG 2).

By assumption (3), the rate of nutrient consumption, \( C \), is proportional to \( V \), so that

\[C = k_2 V,\]

where \( k_2 \) is a second proportionality constant (also positive\(^5\)). The value of \( k_2 \) would depend on the cell metabolism, that is, how quickly it consumes nutrients in carrying out its activities.

By Assumption 1, the cell is spherical, so its surface area, \( S \), and volume \( V \) are:

\[S = 4\pi r^2, \quad V = \frac{4}{3}\pi r^3. \tag{1.2}\]

Putting these facts together leads to the following relationships between nutrient absorption \( A \), consumption \( C \), and cell radius \( r \):

\[A = k_1 (4\pi r^2) = (4\pi k_1)r^2, \quad C = k_2 \left(\frac{4}{3}\pi r^3\right) = \left(\frac{4}{3}\pi k_2\right)r^3.\]

\(^4\)Recall that “A is proportional to B” means that \( A = kB \) where \( k \) is a constant.

\(^5\)From now on, we will simply write \( k_2 > 0 \) is a constant” when we mean this constant to be positive.
Rewriting this relationship as

\[ A(r) = (4\pi k_1) r^2, \quad \text{and} \quad C(r) = \left(\frac{4}{3} \pi k_2\right) r^3. \]  

we observe that that \( A, C \) are simply power functions (LG 3) of the cell radius, \( r \), that is

\[ A(r) = ar^2, \quad C(r) = cr^3 \quad \text{(where} \quad a = 4\pi k_1, \quad c = \frac{4}{3} \pi k_2 \quad \text{are constants).} \]

Importantly, the powers are \( n = 3 \) for consumption and \( n = 2 \) for absorption. The previous discussion of power functions will hence contribute to our analysis of how nutrient balance depends on cell size.

### 1.2.2 Nutrient balance depends on cell size

here we analyze the two power functions for nutrient absorption \( A(r) \) and consumption \( C(r) \) rates as functions of cell radius \( r \) in Eqs. (1.3). We first ask whether absorption or consumption of nutrients dominates for small, medium, or large cells.

**Example 1.3** Is the absorption rate or the consumption rate greater for small cells? For large cells? For what cell size are the two rates balanced?  

**Solution:** For small \( r \), the power function with the lower power of \( r \) (namely \( A(r) \)) dominates, but for very large values of \( r \), the power function with the higher power (\( C(r) \)) dominates. The two rates “balance” (and their graphs intersect) when

\[ A(r) = C(r) \quad \Rightarrow \quad \left(\frac{4}{3} \pi k_2\right) r^3 = (4\pi k_1)r^2. \]

A trivial solution is \( r = 0 \). If \( r \neq 0 \), then, cancelling a factor of \( r^2 \) from both sides,

\[ r = \frac{k_1}{k_2}. \]

Absorption and consumption rates are equal for cells of this size. It follows that for smaller cells, absorption \( A \approx r^2 \) is the dominant process, while for large cells, consumption rate \( C \approx r^3 \) dominates. We conclude that cells larger than the critical size \( r = 3k_1/k_2 \) will be unable to keep up with the nutrient demand, and will not survive since consumption overtakes absorption of nutrients.

Using the above simple geometric argument, we deduced that cell size has strong implications on its ability to absorb nutrients or oxygen quickly enough to feed itself. For these reasons, cells larger than some maximal size (roughly 1 mm in diameter) rarely occur.
1.2. How big can a cell be? A model for nutrient balance

1.2.3 Even and odd power functions

So far, we have considered power functions $y = x^n$ with $x > 0$. Next, allowing the independent variable $x$ to take both positive and negative values brings up some new ideas, including symmetry properties.

Power functions with an even power, $y = x^2, y = x^4, y = x^6$, etc., shown in panel Fig. 1.3a, are symmetric about the $y$ axis. Odd power functions, $y = x, y = x^3, y = x^5$ (Fig. 1.3 b) are symmetric when rotated through 180° about the origin. We adopt the term even function and odd function to describe such symmetry properties. More formally,

\[ f(-x) = f(x) \quad \Rightarrow \quad f \text{ is an even function,} \]

\[ f(-x) = -f(x) \quad \Rightarrow \quad f \text{ is an odd function} \]

Many functions are not symmetric at all, and are neither even nor odd. See Appendix C for further details.

**Example 1.4** Show that the function $y = g(x) = x^2 - 3x^4$ is an even function

**Solution:** For $g$ to be an even function, it should satisfy $g(-x) = g(x)$. Let us calculate $g(-x)$ and see if this requirement holds. We find that

\[ g(-x) = (-x)^2 - 3(-x)^4 = x^2 - 3x^4 = g(x). \]

This solution is not interesting biologically, but we should not forget it in mathematical analysis of such problems.
Here we have used the fact that \((-x)^n = (-1)^nx^n\), and that when \(n\) is even, \((-1)^n = 1\).

All power functions are continuous and unbounded: For \(x \to \infty\) both even and odd power functions satisfy \(y = x^n \to \infty\). For \(x \to -\infty\), odd power functions tend to \(-\infty\). Odd power functions are one-to-one: that is, each value of \(y\) is obtained from a unique value of \(x\) and vice versa. This is not true for even power functions (Fig 1.3a): for example, \(y = 1\) is obtained by evaluating the function \(y = x^2\) at either \(x = 1\) or \(x = -1\), and every other positive value of \(y\) is similarly obtained by evaluating a given power function at a positive or a negative value of \(x\). From Fig 1.3 we see that all power functions go through the point \((0, 0)\). Even power functions have a local minimum at the origin whereas odd power functions do not.

**Definition 1.5 (Local Minimum).** A local minimum of a function \(f(x)\) is a point \(x_{\text{min}}\) such that the value of \(f\) is larger at all sufficiently close points. Formally, \(f(x_{\text{min}} \pm \epsilon) > f(x_{\text{min}})\) for \(\epsilon\) small enough.

### 1.3 Sustainability and energy balance on Earth

The sustainability of life on Planet Earth depends on a fine balance between the temperature of its oceans and land masses and the ability of life forms to tolerate climate change. As a followup to our model for nutrient balance, we briefly introduce a simple energy balance model to track incoming and outgoing energy and to determine a rough estimate for the Earth’s temperature. We use the following basic facts:

1. Energy input from the sun to Earth given the Earth’s radius \(r\) can be approximated as

\[
E_{\text{in}} = (1 - a)S\pi r^2, \tag{1.4}
\]

where \(S\) is incoming radiation energy per unit area (also called the solar constant) and \(0 \leq a \leq 1\) is the fraction of that energy reflected. \(a\) is also called the albedo, and depends on cloud cover, and other aspects of the planet (such as percent forest, snow, desert, and ocean).

2. Energy lost from Earth due to radiation into space depends on the current temperature of the Earth \(T\), and is approximated as

\[
E_{\text{out}} = 4\pi r^2\epsilon\sigma T^4, \tag{1.5}
\]

where \(\epsilon\) is the emissivity of the Earth’s atmosphere, which represents the Earth’s tendency to emit radiation energy. This constant depends on cloud cover, water vapour as well as on greenhouse gas concentration in the atmosphere\(^7\) \(\sigma\) is a physical constant (the Stephan-Bolzmann constant) which is fixed for the purpose of our discussion.

**Example 1.6 (Energy expressions are power functions)** Explain in what sense the two forms of energy above can be viewed as power functions, and what types of power functions they represent.  

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\(^7\)Greenhouse gases include carbon dioxide, and methane.
1.4. Combining power functions: first steps in graph sketching

Solution: Both $E_{\text{in}}$ and $E_{\text{out}}$ depend on Earth’s radius as the power $\sim r^2$. However, since this radius is a constant, it will not be fruitful to consider it as an interesting variable for this problem (unlike the cell size example we previously discussed). However, we note that $E_{\text{out}}$ depends on temperature $T$ as $\sim T^4$. (We might also select the albedo as a variable and in that case, we note that $E_{\text{in}}$ depends linearly on the albedo $a$.)

Example 1.7 (Energy equilibrium for the Earth) Explain how the facts above can be used to determine the equilibrium temperature of the Earth, that is, the temperature at which the incoming and outgoing radiation energies are balanced.

Solution: The Earth will be at equilibrium when

$$E_{\text{in}} = E_{\text{out}} \Rightarrow (1 - a)S\pi r^2 = 4\pi r^2 \epsilon \sigma T^4.$$ 

We observe that the factors $\pi r^2$ cancel, and we obtain an equation that can be solved for the temperature $T$. (See Exercise 21) It is instructive to examine how this temperature depends on the constants in the problem, and how it is affected by cloud cover and greenhouse gas level. We discuss these issues in the same exercise.

1.4 Combining power functions: first steps in graph sketching

Based on the familiarity gained with power functions, we now discuss functions made up of such components. In particular, we extend the discussion to polynomials (sums of power functions) and rational functions (ratios of such functions). We also develop an important skill in sketching graphs of these functions; this skill will prove of great value throughout this course.

Learning goals

1. Be able to easily sketch the graph of a simple polynomial of the form $y = ax^n + bx^m$ (Fig. 1.4).

2. Be able to sketch a rational function such as $y = Ax^n/(b + x^m)$.

1.4.1 Sketching a simple (two-term) polynomial

Example 1.8 (Sketching a simple cubic polynomial) Sketch a graph of the polynomial

$$y = p(x) = x^3 + ax.$$ 

How would the sketch change if the constant $a$ changes from positive to negative?
Figure 1.4. The graph of the polynomial $y = p(x) = x^3 + ax$ can be obtained by putting together its two power function components. The cubic “arms” $y \approx x^3$ (top row) dominate for large $x$ (far from the origin), whereas the linear part $y \approx ax$ (middle row) dominate near the origin. When these are smoothly connected (bottom row) we obtain a sketch of the desired polynomial. Shown here are three possibilities, for $a < 0$, $a = 0$, $a > 0$, left to right. The value of $a$ determines the slope of the curve near $x = 0$ and thus also affects presence of a local maximum and minimum (for $a < 0$).

**Solution:** The polynomial in (1.6) has two terms, each one a power function. Let us consider their effects individually. Near the origin, for $x \approx 0$ the term $ax$ dominates so that, close to $x = 0$, the function behaves as

$$y \approx ax.$$  

This is a straight line with slope $a$. Hence, near the origin, if $a > 0$ we would see a line with positive slope, whereas if $a < 0$ the slope of the line should be negative. Far away from the origin, the cubic term dominates, so

$$y \approx x^3$$

at large (positive or negative) $x$ values. Figure 1.4 illustrates these ideas. In column (a) we see the behaviour of $y = p(x) = x^3 + ax$ for large $x$, in (b) for small $x$. Column (c) shows the graph for an intermediate range. We might notice that for $a < 0$, the graph has a local minimum as well as a local maximum. The simple arguments used above already lead us to a fairly reasonable sketch of the function in (1.6). We can add further details using algebra to find zeros.
Example 1.9 (Zeros) Find the places at which the polynomial (1.6) crosses the $x$ axis, that is, find the zeros of the function $y = x^3 + ax$. 

**Solution:** The zeros of the polynomial can be found by setting

$$y = p(x) = 0 \implies x^3 + ax = 0 \implies x^3 = -ax.$$ 

The above equation always has a solution $x = 0$, but if $x \neq 0$, we can cancel and obtain

$$x^2 = -a.$$ 

This would have no solutions if $a$ is a positive number, so that in that case, the graph crosses the $x$ axis only once, at $x = 0$, as shown in Figure 1.4. If $a$ is negative, then the minus signs cancel, so the equation can be written in the form

$$x^2 = |a|$$

and we would have two new zeros at

$$x = \pm \sqrt{|a|}.$$ 

For example, if $a = -1$ then the function $y = x^3 - x$ has zeros at $x = 0, 1, -1$.

Example 1.10 (A more general case) Explain how you would use the ideas of Example 1.8 to sketch the polynomial $y = p(x) = ax^n + bx^m$. Without loss of generality, you may assume that $n > m \geq 1$ are integers.

**Solution:** As in Example 1.8, this polynomial has two terms that dominate at different ranges of the independent variable. Close to the origin, $y \approx bx^m$ (since $m$ is the lower power) whereas for large $x$, $y \approx ax^n$. The full behaviour is obtained by smoothly connecting these pieces of the graph. Finding zeros can refine the graph. Some examples of this type are discussed in the Exercises.

The reasoning used here is an important first step in sketching a polynomial. Later in this course we develop specialized methods to find zeros of more complicated functions (using an approximation called **Newton’s method**). We will also apply calculus tools to determine points at which the function attains local maxima or minima (called **critical points**), and how it behaves asymptotically, for large positive or negative values of $x$. The elementary steps described here will remain useful in later work as a quick approach for visualizing the overall shape of a graph.

### 1.4.2 Sketching a simple rational function

We use similar reasoning to consider the graphs of simple rational functions. A **rational function** is a function that can be written as

$$y = \frac{p_1(x)}{p_2(x)},$$

where $p_1(x)$ and $p_2(x)$ are polynomials.
Example 1.11 (A rational function) Sketch the graph of the rational function

\[ y = \frac{Ax^n}{a^n + x^n}, \quad x \geq 0. \quad (1.7) \]

What properties of your sketch depend on the power \( n \)? What would the graph look like for \( n = 1, 2, 3 \)?

Solution: We can break up the process of understanding this function into the following steps:

- The graph of the function (1.7) goes through the origin. (At \( x = 0 \), we see that \( y = 0 \).)
- For very small \( x \), (i.e., \( x \ll a \)) we can approximate the denominator by the constant term \( a^n + x^n \approx a^n \), since \( x^n \) is negligible by comparison, so that

  \[ y \approx \frac{Ax^n}{a^n} \approx \left( \frac{A}{a^n} \right) x^n \quad \text{for small } x. \]

  This means that near the origin, the graph looks like a power function, \( Cx^n \) (where \( C = A/a^n \)).
- For large \( x \), i.e. \( x >> a \), we have \( a^n + x^n \approx x^n \) since \( x \) overtakes and dominates over the constant \( a \), so that

  \[ y \approx \frac{Ax^n}{x^n} = A \quad \text{for large } x. \]

  This reveals that the graph has a horizontal asymptote \( y = A \) at large values of \( x \).
- Since the function behaves like a simple power function close to the origin, we conclude directly that the higher the value of \( n \), the flatter its graph near 0. Further, large \( n \) means sharper rise to the eventual asymptote.

The results are displayed in Fig. 1.5.

1.5 Rate of an enzyme-catalyzed reaction

Rational functions introduced in Example 1.11 often play a role in biochemistry. Here we discuss two important examples and the contexts in which they appear. In both cases, we consider the initial rise of the function as well as its eventual saturation.

Learning goals

1. Understand the connection between Michaelis-Menten kinetics in biochemistry and rational functions described in Section 1.4.2.
2. Be able to interpret properties of a graph such as Fig. 1.7 in terms of properties of an enzyme-catalyzed reactions.
1.5. Rate of an enzyme-catalyzed reaction

Figure 1.5. The rational functions (1.7) with \( n = 1, 2, 3 \) are compared on this graph. Close to the origin, the function behaves like a power function, whereas for large \( x \) there is a horizontal asymptote at \( y = A \). As \( n \) increases, the graph becomes flatter close to the origin, and steeper in its rise to the asymptote.

1.5.1 Saturation and Michaelis-Menten kinetics

Biochemical reactions are often based on the action of proteins known as enzymes that catalyze many reactions in living cells. Shown in Fig. 1.6 is a typical scheme. The enzyme \( E \) binds to its substrate \( S \) to form a complex \( C \). The complex then breaks apart into a product, \( P \), and an enzyme molecule that can repeat its action again. Generally, the substrate is much more plentiful than the enzyme.

\[ E + S \overset{k_1}{\underset{k_{-1}}{\rightleftharpoons}} C \overset{k_2}{\rightarrow} E + P \]

Figure 1.6. An enzyme (catalytic protein) is shown binding to a substrate molecule (circular dot) and then processing it into a product (star shaped molecule).

In the context of this example, \( x \) represents the concentration of substrate in the reaction mixture. The speed of the reaction, \( v \), (namely the rate at which product is formed) depends on \( x \). But the relationship is not linear, as shown in Fig. 1.7(a). In fact, this relationships, known as Michaelis Menten kinetics, has the form

\[ \text{speed of reaction} = v = \frac{Kx}{k_n + x}, \quad (1.8) \]

where \( K, k_n > 0 \) are positive constants that are specific to the enzyme and the experimental conditions.

Equation (1.8) is a rational function. Since \( x \) is a concentration, it must be a positive quantity, so we restrict attention to \( x \geq 0 \). The expression in (1.8) is a special case of the rational functions explored in Example 1.11, where \( n = 1, A = K, a = k_n \). In the
Figure 1.7. (a): The graph of reaction speed, \( v \), versus substrate concentration, \( x \) in an enzyme-catalyzed reaction, as in Eqn. 1.8. This behaviour is called Michaelis-Menten kinetics. Note that the graph at first rises almost like a straight line, but then it curves and approaches a horizontal asymptote. This graph tells us that the speed of the enzyme cannot exceed some fixed level, i.e. it cannot be faster than \( K \). (b): Hill function kinetics, from Eqn. (1.7), with \( A = 3, a = 1 \) and Hill coefficient \( n = 1, 2, 3 \). See also Fig 1.5 for an analysis of the shape of this graph.

left panel of Fig. 1.7, we used graphics software to plot this function for specific values of \( K, k_n \). The following observations can be made

1. The graph of (1.8) goes through the origin. Indeed, when \( x = 0 \) we have \( v = 0 \).

2. Close to the origin, the graph “looks like” a straight line. We can see this by considering values of \( x \) that are much smaller than \( k_n \). Then the denominator \((k_n + x)\) is well approximated by the constant \( k_n \). Thus, for small \( x \), \( v \approx (K/k_n)x \). Thus for small \( x \) the graph resembles a straight line through the origin with slope \( (K/k_n) \).

3. For large \( x \), there is a horizontal asymptote. A similar argument for \( x \gg k_n \), verifies that \( v \) is approximately constant at large enough \( x \).

Michaelis-Menten kinetics represents a relationship in which **saturation** occurs: the speed of the reaction at first increases as substrate concentration \( x \) is raised, but the enzymes saturate and operate at a fixed constant speed \( K \) as more and more substrate is added.

It is worth pointing out the units of terms in (1.8). \( x \) carries units of concentration (e.g. nano Molar written nM, which means \( 10^{-9} \) Moles per litre), \( v \) carries units of concentration over time (e.g. nM min\(^{-1}\)), and \( k_n \) must have same units as \( x \). (Only quantities with identical units can be added or compared!) The units on the two sides of the relationship (1.8) have to balance too, meaning that \( K \) must have the same units as the speed of the reaction, \( v \).
1.6. Analysis versus computational tools: two sides of a coin

1.5.2 Hill functions

The Michaelis-Menten kinetics we discussed above fit into a broader class of Hill functions, which are rational functions of the form shown in Eqn. (1.7) with \( n > 1 \) and \( A, a > 0 \). This function is often referred to as a Hill function with coefficient \( n \), (although the “coefficient” is actually a power in terms of the terminology used in this chapter). Hill functions occur when an enzyme-catalyzed reaction benefits from cooperativity of a multi-step process. For example, the binding of the first substrate molecule may enhance the binding of a second.

Michaelis Menten kinetics coincides with a Hill function for \( n = 1 \). In biochemistry, expressions of the form (1.7) with \( n > 1 \) are often denoted “sigmoidal” kinetics. Several such functions are plotted in Fig. 1.7(b). We have already examined the shapes of these functions in Example 1.11.

All Hill functions have a horizontal asymptote \( y = A \) at large values of \( x \). If \( y \) is the speed of a chemical reaction (analogous to the variable we labeled \( v \) on the left panel), then \( A \) is the “maximal rate” or “maximal speed” of the reaction. Since the Hill function behaves like a simple power function close to the origin, the higher the value of \( n \), the flatter is its graph near 0. and the sharper the rise to the eventual asymptote. Hill functions with large \( n \) are often used to represent “switch-like” behaviour in genetic networks or biochemical signal transduction pathways.

The constant \( a \) is sometimes called the “half-maximal activation level” for the following reason: When \( x = a \) then

\[
v = \frac{Aa^n}{a^n + a^n} = \frac{Aa^2}{2a^2} = \frac{A}{2}.
\]

This shows that the level \( x = a \) leads to a reaction speed of \( A/2 \) which is half of the maximal possible rate.

1.6 Analysis versus computational tools: two sides of a coin

Sections 1.5 and 1.4.2 illustrate the fact that mathematical understanding can be gained in a variety of ways. Whereas in Section 1.5 we used reasoning and geometric analysis to sketch graphs of interest, in Section 1.4.2 we relied on software to graph the same functions. The two approaches complement one another: one helps to anticipate the shape of the function, while the other provides greater accuracy. Such complementary approaches will be used often in this course. Rough sketches will supplement the more precise graphing based on calculus, while software will provide computational support for calculations that are otherwise tedious or repetitive.

1.7 For Further Study

Sections G.A and G.B of Appendix G describes two additional topics related to the material in this chapter.
Exercises

1.1. **Power functions:** Consider the power function

\[ y = ax^n, \quad -\infty < x < \infty \]

Explain verbally (or using a sketch) how the shape of the function changes when the coefficient \( a \) increases or decreases (for fixed \( n \)). How is this change in shape different from the shape change that results from changing the power \( n \)?

1.2. **Simple transformations:** Consider the graphs of the simple functions \( y = x, y = x^2, \) and \( y = x^3 \). What happens to each of these graphs when the functions are transformed as follows:

(a) \( y = Ax, y = Ax^2, \) and \( y = Ax^3 \) where \( A > 1 \) is some constant.

(b) \( y = x + a, y = x^2 + a, \) and \( y = x^3 + a \) where \( a > 0 \) is some constant.

(c) \( y = (x - b)^2, \) and \( y = (x - b)^3 \) where \( b > 0 \) is some constant.

1.3. **Simple sketches:** Sketch the graphs of the following functions:

(a) \( y = x^2 \),

(b) \( y = (x + 4)^2 \)

(c) \( y = a(x - b)^2 + c \) for the case \( a > 0, b > 0, c > 0 \).

(d) Comment on the effects of the constants \( a, b, c \) on the properties of the graph of \( y = a(x - b)^2 + c \).

1.4. **Sketching simple polynomials:** Use arguments from Section 1.4 to sketch graphs of the following simple polynomials:

(a) \( y = 2x^5 - 3x^2 \),

(b) \( y = x^3 - 4x^5 \).

1.5. **Finding points of intersection(I):**

(a) Consider the two functions \( f(x) = 3x^2 \) and \( g(x) = 2x^5 \). Find all points of intersection of these functions.

(b) Repeat the calculation for the two functions \( f(x) = x^3 \) and \( g(x) = 4x^5 \).

Observe that finding these points of intersection is equivalent to calculating the zeros of the functions in Problem 4.

1.6. **Qualitative sketching skills:**

(a) Sketch the graph of the function \( y = ax - x^5 \) for positive and negative values of the constant \( a \). Comment on behaviour close to zero and far away from zero.

(b) What are the zeros of this function and how does this depend on \( a \)?

(c) For what values of \( a \) would you expect that this function would have a local maximum (“peak”) and a local minimum (“valley”)?
1.7. **Finding points of intersection (II):** Consider the two functions \( f(x) = Ax^n \) and \( g(x) = Bx^m \). Suppose \( m > n > 1 \) are integers, and \( A, B > 0 \). Determine the values of \( x \) at which the values of the functions are the same. Are there two places of intersection or three? How does this depend on the integer \( m - n \)? (Remark: The point (0,0) is always an intersection point. Thus, we are asking when there is only one more and when there are two more intersection points. See Problem 5 for a simple example of both types.)

1.8. **More intersection points:** Find the intersection of each pair of functions.
   (a) \( y = \sqrt{x}, y = x^2 \)
   (b) \( y = -\sqrt{x}, y = x^2 \)
   (c) \( y = x^2 - 1, \frac{x^2}{4} + y^2 = 1 \)

1.9. **Crossing the \( x \) axis:** Answer the following problem by solving for \( x \) in each case. Find all values of \( x \) for which the following functions cross the \( x \) axis (also called zeros of the function, or roots of the equation \( f(x) = 0 \)).
   (a) \( f(x) = I - \gamma x \), where \( I, \gamma \) are positive constants.
   (b) \( f(x) = I - \gamma x + \epsilon x^2 \), where \( I, \gamma, \epsilon \) are positive constants. Are there cases where this function does not cross the \( x \) axis?
   (c) In the case where the root(s) exist in part (b), are they positive, negative or of mixed signs?

1.10. **Crossing the \( x \) axis, continued:** Answer Problem 9 by sketching a rough graph of each of the functions in parts (a-b) and using these sketches to answer the question of how many real roots there can be and where they are located (on the positive or negative \( x \) axis). **Note:** This problem provides very important qualitative analysis skills that will become useful in later applications.

1.11. **Power functions:** Consider the functions \( y = x^n, y = x^{1/n}, y = x^{-n} \), where \( n \) is an integer (\( n = 1, 2 \ldots \)) Which of these functions increases most steeply for values of \( x \) greater than 1? Which decreases for large values of \( x \)? Which functions are not defined for negative \( x \) values? Compare the values of these functions for \( 0 < x < 1 \). Which of these functions are not defined at \( x = 0 \)?

1.12. **Roots of a quadratic:** Find the range of \( m \) such that the equation \( x^2 - 2x - m = 0 \) has two unequal roots.

1.13. **Rational Functions:** In support of Learning Goal 2 of Section 1.4, describe the shape of the graph of the function \( y = Ax^n/(b + x^m) \) in two cases: (a) \( n > m \) and (b) \( m > n \).

1.14. **Power functions with negative powers:** Consider the function

\[
f(x) = \frac{A}{x^a}\]

where \( A > 0, a > 1 \), with \( a \) an integer. This is the same as the function \( f(x) = Ax^{-a} \), which is a power function with a negative power.
   (a) Sketch a rough graph of this function for \( x > 0 \).
   (b) How does the function change if \( A \) is increased?
(c) How does the function change if $a$ is increased?

1.15. **Intersections of functions with negative powers:** Consider two functions of the form

$$f(x) = \frac{A}{x^a}, \quad g(x) = \frac{B}{x^b}.$$ 

Suppose that $A, B > 0$, $a, b > 1$ and that $A > B$. Determine where these functions intersect for positive $x$ values.

1.16. **Zeros of polynomials:** Find all real zeros of the following polynomials:

(a) $x^3 - 2x^2 - 3x$

(b) $x^5 - 1$

(c) $3x^2 + 5x - 2$.

(d) Find the points of intersection of the functions $y = x^3 + x^2 - 2x + 1$ and $y = x^3$.

1.17. **Inverse functions:** The functions $y = x^3$ and $y = x^{1/3}$ are inverse functions.

(a) Sketch both functions on the same graph for $-2 < x < 2$ showing clearly where they intersect.

(b) The tangent line to the curve $y = x^3$ at the point (1,1) has slope $m = 3$, whereas the tangent line to $y = x^{1/3}$ at the point (1,1) has slope $m = 1/3$. Explain the relationship of the two slopes.

1.18. **Properties of a cube:** The volume $V$ and surface area $S$ of a cube whose sides have length $a$ are given by the formulae

$$V = a^3, \quad S = 6a^2.$$ 

Note that these relationships are expressed in terms of power functions. The independent variable is $a$, not $x$. We say that “$V$ is a function of $a$” (and also “$S$ is a function of $a$”).

(a) Sketch $V$ as a function of $a$ and $S$ as a function of $a$ on the same set of axes. Which one grows faster as $a$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $a$? Sketch a graph of $\frac{V}{S}$ as a function of $a$.

(c) The formulae above tell us the volume and the area of a cube of a given side length. But suppose we are given either the volume or the surface area and asked to find the side. Find the length of the side as a function of the volume (i.e. express $a$ in terms of $V$). Find the side as a function of the surface area. Use your results to find the side of a cubic tank whose volume is 1 litre (1 litre $= 10^3$ cm$^3$). Find the side of a cubic tank whose surface area is 10 cm$^2$.

1.19. **Properties of a sphere:** The volume $V$ and surface area $S$ of a sphere of radius $r$ are given by the formulae

$$V = \frac{4\pi r^3}{3}, \quad S = 4\pi r^2.$$
Note that these relationships are expressed in terms of power functions with constant multiples such as $4\pi$. The independent variable is $r$, not $x$. We say that “$V$ is a function of $r$” (and also “$S$ is a function of $r$”).

(a) Sketch $V$ as a function of $r$ and $S$ as a function of $r$ on the same set of axes. Which one grows faster as $r$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $r$? Sketch a graph of $\frac{V}{S}$ as a function of $r$.

(c) The formulae above tell us the volume and the area of a sphere of a given radius. But suppose we are given either the volume or the surface area and asked to find the radius. Find the radius as a function of the volume (i.e. express $r$ in terms of $V$). Find the radius as a function of the surface area. Use your results to find the radius of a balloon whose volume is 1 litre. (1 litre = $10^3$ cm$^3$). Find the radius of a balloon whose surface area is 10 cm$^2$.

1.20. **The size of cell:** Consider a cell in the shape of a thin cylinder (length $L$ and radius $r$). Assume that the cell absorbs nutrient through its surface at rate $k_1 S$ and consumes nutrients at rate $k_2 V$ where $S, V$ are the surface area and volume of the cylinder. Here we assume that $k_1 = 12 \mu$M $\mu$m$^{-2}$ per min and $k_2 = 2 \mu$M $\mu$m$^{-3}$ per min. (Note: $\mu$M is $10^{-6}$ moles, $\mu$m is $10^{-6}$ meters.) Use the fact that a cylinder (without end-caps) has surface area $S = 2\pi r L$ and volume $V = \pi r^2 L$ to determine the cell radius such that the rate of consumption exactly balances the rate of absorption. What do you expect happens to cells with a bigger or smaller radius? How does the length of the cylinder affect this nutrient balance?

1.21. **Energy equilibrium for Earth:** This problem focuses on Earth’s temperature, climate change, and sustainability.

(a) Complete the calculation for Example 1.7 by solving for the temperature $T$ of the Earth at which incoming and outgoing radiation energies balance.

(b) Assume that greenhouse gasses decrease the emissivity $\epsilon$ of the Earth’s atmosphere. Explain how this would affect the Earth’s temperature.

(c) Explain how the size of the Earth affects its energy balance according to the model.

(d) Explain how the albedo $a$ affects the Earth’s temperature.

1.22. **Allometric relationship:** Properties of animals are often related to their physical size or mass. For example, the metabolic rate of the animal ($R$), and its pulse rate ($P$) may be related to its body mass $m$ by the approximate formulae $R = A m^b$ and $P = C m^d$, where $A, C, b, d$ are positive constants. Such relationships are known as *allometric* relationships.

(a) Use these formulae to derive a relationship between the metabolic rate and the pulse rate (Hint: eliminate $m$).

(b) A similar process can be used to relate the Volume $V = (4/3)\pi r^3$ and surface area $S = 4\pi r^2$ of a sphere to one another. Eliminate $r$ to find the corresponding relationship between volume and surface area for a sphere.
1.23. **Rate of a very simple chemical reaction:** Here we consider a chemical reaction that does not saturate, and consider the simple linear relationship between reaction speed and reactant concentration. A chemical is being added to a mixture and is used up by a reaction that occurs in that mixture. The rate of change of the chemical, (also called “the rate of the reaction”) \( v \) (in units of M/sec where M stands for Molar, which is the number of moles per litre) is observed to follow a relationship \( v = a - bc \) where \( c \) is the reactant concentration (in units of M) and \( a, b \) are positive constants. (Note that here \( v \) is considered to be a function of \( c \), and moreover, the relationship between \( v \) and \( c \) is assumed to be linear.)

(a) What units should \( a \) and \( b \) have to make this equation consistent? (Remember: in an equation such as \( v = a - bc \), each of the three terms must have the same units. Otherwise, the equation would not make sense.)

(b) Use the information in the graph shown in Figure 1.8 to find the values of \( a \) and \( b \). (To do so, you should find the equation of the line in the figure, and compare it to the relationship \( v = a - bc \).)

(c) What is the rate of the reaction when \( c = 0.005 \) M?

![Figure 1.8. Figure for problem 23](image)

1.24. **Michaelis-Menten kinetics:** Consider the Michaelis-Menten kinetics where the speed of an enzyme-catalyzed reaction is given by \( v = \frac{Kx}{(k_n + x)} \).

(a) Explain the statement that “when \( x \) is large there is a horizontal asymptote” and find the value of \( v \) to which that asymptote approaches.

(b) Determine the reaction speed when \( x = k_n \) and explain why the constant \( k_n \) is sometimes called the “half-max” concentration.

1.25. **A polymerization reaction:** Consider the speed of a polymerization reaction shown in Figure 1.9. Here the rate of the reaction is plotted as a function of the substrate concentration. (The experiment concerned the polymerization of actin, an important structural component of cells; data from [12].) The experimental points are shown as dots, and a Michaelis-Menten curve has been drawn to best fit these points. Use the data in the figure to determine approximate values of \( K \) and \( k_n \) in the two treatments shown.
1.26. **Hill functions**: Hill functions are sometimes used to represent a biochemical “switch”, that is a rapid transition from one state to another. Consider the Hill functions

\[ y_1 = \frac{x^2}{1 + x^2}, \quad y_2 = \frac{x^5}{1 + x^5}, \]

(a) Where do these functions intersect?
(b) What are the asymptotes of these functions?
(c) Which of these functions increases fastest near the origin?
(d) Which is the sharpest “switch” and why?

1.27. **Transforming a Hill function to a linear relationship**: A Hill function is a non-linear function. But if we redefine variables, we can transform it into a linear relationship. The process is analogous to transforming Michaelis-Menten kinetics into a Lineweaver-Burke plot. Determine how to define appropriate variables \( X \) and \( Y \) (in terms of the original variables \( x \) and \( y \)) so that the Hill function \( y = \frac{Ax^3}{a^3 + x^3} \) is turned into a linear relationship between \( X \) and \( Y \). Then indicate how the slope and intercept of the line are related to the original constants \( A, a \) in the Hill function.

1.28. **Hill function and sigmoidal chemical kinetics**: It is known that the rate \( v \) at which a certain chemical reaction proceeds depends on the concentration of the reactant \( c \) according to the formula

\[ v = \frac{Kc^2}{a^2 + c^2} \]

where \( K, a \) are some constants. When the chemist plots the values of the quantity \( 1/v \) (on the “\( y \)” axis) versus the values of \( 1/c^2 \) (on the “\( x \)” axis”), she finds that the points are best described by a straight line with \( y \)-intercept 2 and slope 8. Use this result to find the values of the constants \( K \) and \( a \).
1.29. **Linweaver-Burke plots**: Shown in the Figure (a) and (b) are two Linweaver Burke plots. By noting properties of these figures comment on the comparison between the following two enzymes:

(a) Enzyme (1) and (2).

(b) Enzyme (1) and (3).

![Linweaver-Burke plots](image)

**Figure 1.10. Figure for problem 29**

1.30. **Michaelis Menten Enzyme kinetics**: The rate of an enzymatic reaction according to the *Michaelis Menten Kinetics* assumption is

\[
v = \frac{Kc}{k_n + c},
\]

where \(c\) is concentration of substrate (shown on the \(x\) axis) and \(v\) is the reaction speed (given on the \(y\) axis). Consider the data points given in the table below:

<table>
<thead>
<tr>
<th>Substrate conc</th>
<th>nM</th>
<th>Reaction speed</th>
<th>nM/min</th>
<th>c</th>
<th>v</th>
<th>5.</th>
<th>10.</th>
<th>20.</th>
<th>40.</th>
<th>50.</th>
<th>100.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>0.068</td>
<td>0.126</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0.218</td>
<td>0.345</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.39</td>
<td>0.529</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Convert this data to a Linweaver-Burke (linear) relationship. Plot the transformed data values on a graph or spreadsheet, and estimate the slope and \(y\)-intercept of the line you get. Use these results to find the best estimates for \(K\) and \(k_n\).

1.31. **Spacing in a school of fish**: According to the biologist Breder [2], two fish in a school prefer to stay some specific distance apart. Breder suggested that the fish that are a distance \(x\) apart are attracted to one another by a force \(F_A(x) = A/x^a\) and repelled by a second force \(F_R(x) = R/x^r\), to keep from getting too close. He found the preferred spacing distance (also called the *individual distance*) by determining the value of \(x\) at which the repulsion and the attraction exactly balance. Find the *individual distance* in terms of the quantities \(A, R, a, r\) (all assumed to be positive constants.)