Exponentials

And more implicit differentiation and related rates
Results of the Midterm Test
Total scores

Students: 895  Median: 35  Mean: 34.57  Std. Dev: 9.269
Total scores Section 110
Mean: 37.5
Total scores Section 110
Mean: 37.5
2. Which of the following describes the point (1, 1) in the function \( f(x) = x^4 - 4x^3 + 4x^2 \)?

A. local minimum
B. local maximum
C. inflection point
D. root
E. none of the above

I’m gonna need a 2\(^{nd}\) derivative to check these out!
MC: Q3

. The quantity 2000 represents the

A. value of \( C(t) \) at \( t = 3 \).
B. difference \( C(3) - C(0) \).
C. average rate of change of \( C(t) \) from \( t = 0 \) to \( t = 3 \).
D. instantaneous rate of change of \( C(t) \) from \( t = 0 \) to \( t = 3 \).
E. slope of the tangent line of \( C(t) \) at \( t = 3 \).
MC: Q4

\[ y = \frac{x}{\sqrt{x^4 + 4}} \]
Sketching skills

\[ y = \frac{x}{\sqrt{x^4 + 4}} \]

Small \( x \)

Large \( x \)
MC: Q5 (spreadsheet problem)

The spreadsheet below will be used to approximate a root of \( f(x) = x^5 + 3x^3 - 7 \) using Newton’s Method with \( x_0 = 1 \). An arrow indicates the content of a cell will be copied down its column. What should go in cells A1, B1, and C1?

Circle the numeral next to the correct entry for each.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>=A1-B1/C1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>↓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>↓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• What goes in cells A1, B1, C1
MC: Q5 scores

Students: 895  Median: 3  Mean: 2.74  Std. Dev: 0.703

We LOVE spreadsheets!!
MC: Q6 Hard

Which of the following statements is true for any differentiable function \( f \)?

A. If the graph of \( f'(x) \) looks like the line with equation \( y = 3x \) when zooming in at \( x = 0 \), then \( f(x) \) has a local minimum at 0.

B. If \( f''(\pi) = 0 \), then the concavity of \( f(x) \) changes at \( \pi \).

C. If \( f'(x) \) is continuous, then \( f''(x) \) exists.

D. If the function \( f(x) \) considered on the interval \( 2 \leq x \leq 3 \) has an absolute minimum over this interval, then \( f'(x) \) has a zero.

E. More than one of the above statements is true.
MC: Q7

Given the derivative, $f'(x)$ shown below:

Which of the following is the original function?
Q8: graphing

Graph the function \[ y = f(x) = \frac{1}{2} x^2 \left( 1 - \frac{1}{3} x \right). \]

Note that \[ f'(x) = x \left( 1 - \frac{1}{2} x \right) \]
\[ f''(x) = 1 - x \]
Solution:

- Min at origin; Max at $x = 2$; Inflection at $x = 1$,
- Root at $x=3$, additional features (height, etc)
Q9: limits

Shown below is the graph $y = f(x)$. Compute the five limits below, or write “DNE”
Q9: Limits: SOLUTION

\[
\lim_{x \to 0} f(x) = 0
\]
\[
\lim_{x \to -5^+} f(x) = -1 \quad \text{Note } x \text{ goes to } -5 \text{ from the right}
\]
\[
\lim_{x \to 2} f(x) = \text{DNE}
\]
Q9: Limits: SOLUTION

\[ \lim_{{h \to 0^-}} \frac{f(4+h) - f(4)}{h} = 2 \]

Note $h$ goes to 0 from the left

\[ \lim_{{h \to 0}} \frac{f(h - 3) - f(-3)}{h} = 0 \]
A cell of the bacterium E. coli has the shape of a cylinder with two hemi-spherical caps, as shown below. Consider this shape, with $h$ the height of the cylinder, and $r$ the radius of the cylinder and hemispheres.

**Sphere**

\[ V = \frac{4}{3} \pi r^3 \]
\[ S = 4 \pi r^2 \]

**Cylinder**

\[ V = \pi r^2 h \]
\[ S = 2\pi rh \]
Q 10, cont’d

(a) Find the value of $h$ that leads to the largest volume for a fixed constant surface area, $S = \text{constant}$. (6 points)

(b) Describe or sketch the shape you found in (a). (1 point)

(c) A typical E. coli cell has $h = 1\mu m$ and $r = 0.5\mu m$. Based on your results in (a) and (b), would you agree that E. coli has a shape that maximizes its volume for a fixed surface area? Explain your answer in one sentence.
Steps in the solution: set-up

Total volume:
\[ V = \frac{4}{3}\pi r^3 + \pi r^2 h. \]

Surface area = constant = 
\[ S = 4\pi r^2 + 2\pi rh. \]

Use this to eliminate one variable, e.g. \( h \):
\[ h = \frac{S - 4\pi r^2}{2\pi r}. \]

Goal: maximize
\[ V = \frac{4}{3}\pi r^3 + \pi r^2 \left(\frac{S - 4\pi r^2}{2\pi r}\right). \]
Finding critical points

Derivatives:

\[ V'(r) = -2\pi r^2 + \frac{S}{2} = 0. \]

Critical point:

\[ r = \frac{1}{2} \sqrt{\frac{S}{\pi}} \]

Test:

\[ V''(r) = -4\pi r < 0. \]

\[ \rightarrow \text{Critical point is a local max} \]
What does it tell us?

• We found the radius

\[ r = \frac{1}{2} \sqrt{\frac{S}{\pi}} \]

• Then the height is

\[ h = \frac{S - 4\pi r^2}{2\pi r} = 0 \]

• Optimal shape is a sphere!!

\( h=0 !!! \)
Q10 scores

Students: 895  Median: 0  Mean: 1.04  Std. Dev: 1.746

Marking scheme:
(a) 1 point: finding the equation of the (total) volume
1 point: finding the relationship between \( r \), \( h \), and \( S \)
1 point: getting volume in terms of one variable and \( S \)
1 point: differentiating 1 point: finding the CP
1 point: finding the correct height

(b) 1 point

(c) 1 point
A fox and a hare are running. At time $t = 0$, they are at the same position.
The fox’s velocity is a constant $v$, where $0 < v < 4$.
The hare’s position at time $t$ is given by
\[ H(t) = \frac{12t}{3 + t} \]

(a) What is the position of the fox at time $t$, $F(t)$? (Note it starts at the same place as the hare.) (2 points)

(b) Just after $t = 0$, the hare is ahead of the fox. When do the fox and hare meet again next? (2 points)

(c) When is the distance between the two animals increasing, and when is it decreasing, between $t = 0$ and their meeting from part (b)? (4 points)
Q 11: Solution

• (a) Fox constant velocity $\rightarrow$ position is $v \ t$

• (b) Equate positions ($x_{\text{hare}} = x_{\text{fox}}$) and solve for $t$ (answer will be in terms of $v$)

• (c) Write $D(t) = x_{\text{hare}} - x_{\text{fox}}$ in terms of $t$ and $v$ find $D'(t)$. Find where $D'(t) = 0$ and where $< 0$, $> 0$ to answer where increasing, decreasing.
12. (a) Use a linear approximation to find a reasonable rational number approximating $\sqrt{98}$. Simplify your answer. (4 points)

(b) Let $c > \sqrt{2}$. Use a linear approximation to find a reasonable number approximating $\sqrt{c^2 - 2}$, in terms of $c$. Simplify your answer. (2 points)
“Lessons learned”

• Bombing 1 test does not ruin your career
• With dedicated work and study you can recover and do OK; or you can improve and meet your goals
• Make a list of concepts that you missed
• Look for extra practice problems
• Spend some time every day reviewing, do lots of HARD problems while keeping up with the new material.
Problems from last time
From last time:
Implicit differentiation

Find the slope of the tangent line at the point (1,1) on the curve

\[ x^2 + xy + y^2 = 3 \]
Example:

Find the slope of the tangent line at the point (1,1) on the curve.

\[ x^2 + xy + y^2 = 3 \]

Using the product rule:

\[ 2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = \frac{d^2}{dx} = 0 \]

\[ 2x + y + \frac{dy}{dx} (x + 2y) = 0 \]

\[ \frac{dy}{dx} = \frac{-2x + y}{x + 2y} \]

\[ \frac{dy}{dx} = -\frac{3}{3} = -1 \]

Plug in the point (1,1) at the end.
Consider the circle given by $x^2 + y^2 = 1$, and the parabola

$$y = ax^2 - b.$$ 

For what values of the constants $a, b$ does the parabola touch the circle at two points?
Motivation

https://www.youtube.com/watch?v=ej3ioOneTy8
The spaceship supply problem

A supply space craft has been put into a circular orbit around Earth. A space-station is supposed to meet this craft to pick up the supplies. Design a parabolic orbit for the space-craft that will give it two opportunities to pick up the supplies.

A.k.a. “Help Matt Damon survive Mars”: https://www.youtube.com/watch?v=ej3ioOneTy8
The plan

Circle

\[ x^2 + y^2 = 1 \]

Parabola

\[ y = ax^2 - b \]

circular orbit

Parabolic orbit
(M1) What is the math problem here?

A) Optimize the distance between two curves
B) Relate some rates using a chain rule
C) Linear approximation of the curves
D) Find intersection points of two curves
E) Match slopes and points on two curves
(M2) What should we do?

A  Find points of intersection
B  Find tangent lines
C  Equate slopes of tangents to both curves
D  Both A and B
E  Both A and C
Solution: Set up the problem

Label the point(s) at which the orbits intersect

\[ y = ax^2 - b \]
\[ x^2 + y^2 = 1 \]
Equate slopes of tangent lines

\[
x^2 + y^2 = 1 \quad \quad \quad y = ax^2 - b
\]

Find \( \frac{dy}{dx} \) for both the curves and get the equation that results by setting them equal at the points \((x_0, y_0)\).

What are possible solutions?
Get slopes of tangents to each curve

Implicit differentiation used for slope of circle

slope of parab: \[ \frac{dy}{dx} = 2ax \]

slope of circle: \[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \]
Equate slopes at meeting point

at pt. of tangency $2ax_0 = \frac{-x_0}{y_0}$

There are two possible cases
Case 1

This case is less interesting – it turns out to have one single solution at the tip or the base of the circle:
Case 2

Two intersection points. Find the value of $b$ in terms of $a$ for this to occur! Find a suitable range for $a$. 

\[ y = ax^2 - b \]
Exponential Growth
Exponentials: $e^x$ and other bases
Practice graphing

Sketch the graph of the function \( y = x^2 e^{-x} \)
(1) Graphing

My graph should look like:

\[ y = x^2 e^{-x} \]
If powers and exponentials “fight” for domination .. then

• Exponentials always win!!

• As $x \to \infty$ $x^2 \to \infty$ power term INCREASING

• As $x \to \infty$ $e^{-x} \to 0$ exponential DECREASING

• the product $x^2 e^{-x} \to 0$ DECREASES as $x \to \infty$
The graph looks like:

\[ y = x^2 e^{-x} \]

Find the critical points, and the value at \( x=0 \), small \( x \), and large \( x \).

(See details at end of slides)
(2) The derivative of $e^x$ is $e^x$

What is the derivative of $y = e^{kx}$

(A) $e^{kx}$
(B) $ke^{(k-1)x}$
(C) $ke^{kx}$
(D) $ke^{kx-1}$
(E) $ke^{k(x-1)}$
The derivative of $e^x$ is $e^x$

By the chain rule, the derivative of $y = e^{kx}$ is $\frac{dy}{dx} = k e^{kx}$

Steps shown here ➔

$y = e^{kx}$, let $u = kx$

\[
\frac{dy}{dx} = \frac{de^u}{du} \cdot \frac{du}{dx} = e^u \cdot k = k e^{kx}
\]
Anatomy of the exponential

\[ \exp(-x) \]

\[ \exp(x) \]

UBC Math 102
Exponentials and logarithms are inverse functions
Inverse functions

If $f$ and $g$ are inverse functions then*

$$y = f(x) \iff x = g(y)$$

$(a,b)$ on graph of $f \iff (b,a)$ on graph of $g$

* On the domains of definition
Symmetry properties

If $g(x)$ is an inverse function of $f(x)$ then their graphs are symmetric about line $y=x$
Slopes and derivatives

Slope of $f(x)$ at $(a,b)$: $\Delta y/\Delta x$
Slope of $g(x)$ at $(b,a)$: $\Delta x/\Delta y$

$f'(a) = 1/g'(b)$

Derivatives are reciprocals at the corresponding pts.
Desmos demo:

\[ f(x) = \exp(x) \]

\[ g(x) = \ln(x) \]
Desmos demo: “the chopsticks”

Challenge: see if you can create this demo which shows the tangent lines to \( \ln(x) \) and \( \exp(x) \) at corresponding points, demonstrating that slopes of these inverse functions are reciprocals.
Last time

- We used implicit differentiation to show that for

\[ y = \ln(x) \]

The derivative is

\[ \frac{dy}{dx} = \frac{1}{x} \]
(3) Base conversion

The function $y = a^x$ can also be written as $y = e^{kx}$

(A) $k = a$
(B) $k = e^a$
(C) $k = \ln(e)$
(D) $K = \ln(a)$
(E) Not sure
Use of ln for base conversion

We can use a simple operation to convert any exponential function of the form $y = a^x$ to base $e$:

\[
y = a^x
\]

\[
\ln y = \ln a^x
\]

\[
\ln y = x \ln a
\]

\[
e^{\ln y} = e^{x \ln a}
\]

\[
y = e^{\frac{x \ln a}{k}}
\]

\[
y = e^{x k}
\]

\[
k = \ln(a)
\]
(4) The derivative of $y = a^x$

Based on the conversion, the derivative of $y = a^x$ is

(A) $y' = a^x$
(B) $y' = x a^{x-1}$
(C) $y' = \ln(a) \ a^x$
(D) $y' = \ln(a^x)$
(E) Not sure
The derivative of $y = a^x$

\[
y = a^x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx} (a^x) \quad \text{chain rule}
\]

\[
y = e^{kx} \quad \Rightarrow \quad \frac{dy}{dx} = k e^{kx}
\]

\[
k = \ln(a) = \text{const.}
\]

\[
\frac{d}{dx} (a^x) = \ln(a) a^x
\]
From a previous lecture: derivative of \( y = a^x \) using definition of the derivative:

\[
\frac{da^x}{dx} = \lim_{h \to 0} \frac{(a^{x+h} - a^x)}{h} = \lim_{h \to 0} \frac{(a^x a^h - a^x)}{h} = \lim_{h \to 0} a^x \frac{(a^h - 1)}{h} = a^x \left[ \lim_{h \to 0} \frac{a^h - 1}{h} \right] = c a^x
\]

This thing is a constant.

The weird constant we encountered (in the red rectangle) is actually the same as \( \ln(a) \)
What’s the logarithm good for?
Application of logarithms to data-fitting

• Log transforms can help to visualize data that varies on a wide scale
• Log transforms can help “fit data” by converting exponential or power relationships to linear relationships.
Log transforms: \textit{exponential} $\rightarrow$ \textit{linear}

A relationship of the form $y = C e^{kx}$ can be expressed as a linear relationship between $\ln(y)$ and $x$:

\[
\begin{align*}
    y &= C e^{kx} \\
    \ln(y) &= \ln(C e^{kx}) \\
    &= \ln(C) + \ln(e^{kx}) \\
    \ln(y) &= \ln(C) + kx
\end{align*}
\]
(5) Log transforms: exponential $\rightarrow$ linear

On a plot of $\ln(y)$ vs $x$ \hspace{1cm} \ln(y) = \ln(C) + k \times x

(A) The slope is $C$ and the intercept is $k$
(B) The slope is $k$ and the intercept is $\ln(C)$
(C) The slope is $k$ and the intercept is $C$
(D) The slope is $-k$ and the intercept is $\ln(C)$
(E) Not sure.
Transformed data

On a log plot, an exponential function looks like a straight line!

\[ y = C e^{kx} \]

\[ \ln(y) = \ln(C) + kx \]

slope = \( k \)
Example:

Assignment 8: Problem 6

Determine whether the function whose values are given in the table below could be linear or exponential.

<table>
<thead>
<tr>
<th>x = 0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = 2</td>
<td>12</td>
<td>72</td>
<td>432</td>
<td>2592</td>
</tr>
</tbody>
</table>

Use a spreadsheet: (1) copy and paste the data. Compute ln(y).
Cont’d: plot \( \log(y) \) vs \( x \) and add trend-line

\[
y = 0.7782x + 0.301
\]

Careful! This is actually \( \log(y) \) vs \( x \)!
Log transforms: \textit{power fn} \rightarrow \textit{linear}

A relationship of the form \( y = a \ x^b \) can be expressed as a linear relationship between \( \ln(y) \) and \( \ln(x) \):

\[
\begin{align*}
    y &= a \ x^b \\
    \ln(y) &= \ln(a \ x^b) \\
    &= \ln(a) + \ln(x^b) \\
    &= \ln(a) + b \ln(x)
\end{align*}
\]
(6) Log transforms: power fn $\rightarrow$ linear

On the graph: $\ln(y) = \ln(a) + b \ln(x)$

(A) The slope is $\ln(x)$ and the intercept is $b$
(B) The slope is $\ln(a)$ and the intercept is $b$
(C) The slope is $b$ and the intercept is $\ln(y)$
(D) The slope is $b$ and the intercept is $\ln(a)$
(E) Not sure
Transformed data

On a log-log plot, a power function looks like a straight line!

$$y = a \, x^b$$

slope = $b$
Example: Application to allometry: “mice to elephant curve”

Metabolic rate vs animal size.

Data:

<table>
<thead>
<tr>
<th>animal</th>
<th>body weight (gm)</th>
<th>metabolic rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>mouse</td>
<td>25</td>
<td>1580</td>
</tr>
<tr>
<td>rat</td>
<td>226</td>
<td>873</td>
</tr>
<tr>
<td>rabbit</td>
<td>2200</td>
<td>466</td>
</tr>
<tr>
<td>dog</td>
<td>11700</td>
<td>318</td>
</tr>
<tr>
<td>man</td>
<td>70000</td>
<td>202</td>
</tr>
<tr>
<td>horse</td>
<td>700000</td>
<td>106</td>
</tr>
</tbody>
</table>
Does data fit a power-law?

• Plot \( \ln(y) \) versus \( \ln(x) \)

(1) This helps to visualize the data
(2) Helps to find values for the constants \( a \) and \( b \)
Fit a straight line to the log-log plot
Determine slope and intercept.
Use these to compute $a,b$

$y \approx ax^b,$

Slope $= b = -0.26$

$8.2 = A$
(7) Values of $a$ and $b$

From this line we find that

(A) $a=\ln(8.2)$, $b=-0.26$
(B) $a=-0.26$, $b=8.2$
(C) $a=8.2$, $b=-0.26$
(D) $a=e^{8.2}$, $b=-0.26$
(E) Not sure
Finding $a$ and $b$:

- $A = 8.2 = \ln(a)$ so $a = e^{8.2} = 3640$
- Slope $= b = -0.26$

Hence:

Data fits relationship

$$y = ax^b = 3640x^{-0.26}$$
Exponential time behaviour, half life and doubling times
Exponentially decaying process

Consider the exponential function of time:

\[ y = f(t) = Ce^{-kt} \]

(with k > 0). What is the value of y at time t=0? (we will denote that value by \( y(0) = y_0 \).)
(8) Exponentially decaying process

For the exponential function \( y = f(t) = C \ e^{-kt} \)

(A) \( y(0) = y_0 = C \)
(B) \( y(0) = y_0 = 0 \)
(C) \( y(0) = y_0 = 1 \)
(D) \( y(0) = y_0 = -C \ k \)
(E) \( y(0) = y_0 = C \ e^{-k} \)
Determine the “half-life”

Definition: the half-life is the time at which ½ of the initial amount remains, i.e. \( y(t) = (1/2) y_0 \)
(9) The “half-life”

Definition: the half-life is the time at which \( \frac{1}{2} \) of the initial amount remains, i.e. \( y(t) = \left( \frac{1}{2} \right) y_0 \)

The half life is

(A) \( \frac{C}{2} \)
(B) \( \frac{k}{2} \)
(C) \( \frac{2}{k} \)

(D) \( \frac{\ln(2)}{k} \)
(E) \( \frac{\ln(C)}{2k} \)
Solution:

Given: \( y = Ce^{-kt} \)
\( y(0) = y_0 = C \)

Want: time \( t \) such that \( y = \frac{y_0}{2} = \frac{C}{2} \)

\( \frac{C}{2} = Ce^{-kt} \)

solve for \( t \!:\)

\( 2 = e^{kt} \)

\( ln(2) = ln(e^{kt}) = kt \)

\( \frac{ln(2)}{k} = t \) \( \leftarrow \) half life
Similar Example:

The amount of cesium-137, remaining $t$ years after the Chernobyl nuclear plant exploded can be modeled by the equation

$$y_C(t) = y_0 e^{-0.023t}$$

where $y_0$ is the initial level. What is the half-life? How long would it take for the level of cesium-137 to be reduced to 3% of its initial level?

$$0.03 y_0 = y_0 e^{-0.023t} \quad \leftarrow \text{solve for } t$$
Exponentially increasing process

Consider the exponential function of time:

\[ y = f(t) = Ce^{kt} \]

(with \( k > 0 \)). The time at which \( y \) doubles is called the **doubling time**.

Exercise: show that the doubling time is \( \frac{\ln(2)}{k} \).
Solution:

\[ y = Ce^{kt} \]

When does \( y \) double?

\[ 2C = C e^{kt} \]

\[ 2 = e^{kt} \]

Note that we get the same equation as we did for half-life, so same answer! doubling time = \( \frac{\ln(2)}{k} \)
Units

• (1) An exponent cannot have units. (Thus, for $P(t) = C e^{rt}$, we conclude that $r$ has units of $1$/year).

• (2) $e$ (and therefore $e^x$) does not have dimensions. (Thus, for $P(t) = C e^{rt}$, we can conclude that $C$ has the same units as $P(t)$).
Applications:

Human population is increasing ("exponential growth")
Long Term Food Security
Motivation for the group-quiz this week

- 30% reduction in arable land since 1961
- 9.2 billion people by 2050
- Food prices rising
- Urbanization is increasing

Credit: Aspire Food Group
Source available
Answers

1 A
2 C
3 D
4 C
5 B
6 D
7 D
8 A
9 D
## Exponentials and logarithms (review)

### Properties of $e^x$
- $e^0 = 1$
- $e^{-x} = \frac{1}{e^x}$
- $e^a e^b = e^{a+b}$
- $(e^a)^b = e^{a \cdot b}$
- $\text{as } x \to \infty \quad e^x \to \infty$
- $\text{as } x \to -\infty \quad e^x \to 0$
- $e^x > 0 \quad \text{for all } x$
- $\frac{de^x}{dx} = e^x$

### Properties of natural logs
- $\ln(1) = 0$
- $\ln\left(\frac{1}{A}\right) = -\ln(A)$
- $\ln(AB) = \ln A + \ln B$
- $\ln(A^B) = B \ln(A)$
- $\text{as } x \to \infty \quad \ln(x) \to \infty$
- $\text{as } x \to 0 \quad \ln(x) \to -\infty$
- $\ln(x)$ is not defined for $x < 0$
- $\frac{d\ln(x)}{dx} = \frac{1}{x}$
Problems to test your skills
A midterm problem “puzzler”

Compute the derivative of $y = f(x) = x^x$

Hint: take $\ln$ of both sides
iii. (3 points) Let $f^{-1}$ be the inverse function of $f(x)$. Assume $f(0) = 1$ and $f'(0) = 2$. Find the tangent line $y = mx + b$ to $f^{-1}(x)$ at 1.
Solutions from last time
(10) The graph of $e^x$ looks like which of these?
Find the equation of this tangent line

Tangent line to the curve $y = e^x$ at $x = 0$.

Solution: The slope is

$$\frac{d(e^x)}{dx} = e^x$$

At $x = 0$ slope is $m = e^0 = 1$

TL goes through $(0, e^0) = (0, 1)$

$b = 1$, and equation

$$y = x + 1$$
Practice: conceptual midterm question

Here is the graph of $y = C e^{kt}$ for some constants $C$, $k$, and a tangent line. Use data from the graph to determine $C$ and $k$. 

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Solution:

From the graph we see that

\[ t = 0 \quad y = y \]
\[ y = Ce^{kt} \quad y(0) = C \quad z = y \]

Also

\[ \frac{dy}{dt} = kCe^{kt} \quad y'(0) = kC = \frac{z}{q} \]

But (from the graph), we can determine the slope of the tangent line and hence find \( k \):

\[ \text{Slope} = \frac{y - 0}{0 - 2} = -2 \]
\[ kC = -2 \]
\[ k = \frac{-2}{4} = -\frac{1}{2} \]
Practice graphing

Sketch the graph of the function  \[ y = x^2 e^{-x} \]
Solution

- At $x=0$ $y=0$
- Critical points?

$$y'(x) = \frac{dy}{dx} = 2x e^{-x} - x^2 e^{-x} = 0$$

$$(2x - x^2) e^{-x} = 0$$

- Solve for $x$:

$$(2x - x^2) = 0$$

$$x(2-x) = 0$$

so $x=0, 2$ are critical pts
Solution, contd

- Second derivative

\[ y''(x) = \frac{d}{dx} [(2x - x^2)e^{-x}] \]

\[ = (2-2x)e^{-x} - (2x - x^2)e^{-x} \]

\[ = (2-4x+x^2)e^{-x} \]

At \( x=0 \)  \( y''(0)=2 \) e^0=2>0  concave up (local max)

At \( x=2 \)  \( y''(2)=(2-8+4)=-2 <0 \) concave down (local min)
Solution, contd

- Inflection points (Ips)

\[ y''(x) = (2-4x+x^2)e^{-x} \]

\[ y''(x) \text{ changes sign when } (2-4x+x^2)=0 \]

Then, using the quadratic formula, we find Ips at

\[ x = \frac{2 + \sqrt{2}, 2 - \sqrt{2}}{2} \]
The graph looks like:

\[ y = x^2 e^{-x} \]
Practice Exam question from last time:

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4. Consider the function

\[ y = f(x) = (1 - x^2)e^{-x^2} \]

(a) Find all critical points. (Note: the second derivative is not required in this problem.)

(b) Sketch the graph of this function on the grid provided. (Note that the x axis scale is 
\(-2 < x < 2\) and the y axis scale is \(-1 < y < 1\). Also: it may or may not be helpful to 
use the following facts: \(\sqrt{2} \approx 1.41, e \approx 2.72, 1/e \approx 0.37\, 1/e^2 \approx 0.135\).)
Solution:

(a) The function is

\[ y = f(x) = (1 - x^2)e^{-x^2} \]

Its derivative is

\[ f'(x) = (-2x)e^{-x^2} + (1 - x^2)(-2x)e^{-x^2} = (-2x)(1 + 1 - x^2)e^{-x^2} = (-2x)(2 - x^2)e^{-x^2} \]

The critical points occur when

\[ f'(x) = 0 \]

Since \( e^{-x^2} > 0 \), critical points occur at \( x = 0, \pm \sqrt{2} \).

(b) We also see that \( f(0) = 1, f(1) = f(-1) = 0 \). The function is negative for \( |x| > 1 \). Near the origin, the graph would look like \( y \approx 1 - x^2 \). (This also implies that the critical point at 0 is a local maximum). For \( |x| > \sqrt{2} \), the function becomes less negative. Far away it would decay to zero due to the exponential function. The graph is symmetric.
Solution, cont’d
Practice Exam question for Implicit differentiation from last time

Question 4: [10 points] Consider the curve whose equation is

\[ x^6 - 3xy + y^6 = 1 \]

(a) Find the slope of the tangent line at the point \((1, 0)\) on this curve.

(b) Determine whether the curve is concave up or concave down at the point \((1, 0)\).
Solution:

(a) Find the slope of the tangent line at the point $(1, 0)$ on this curve.

\[ 6x^5 - 3y - 3xy' + 6y^5 y' = 0 \]

\[ (6x^5 - 3y) + y' (6y^5 - 3x) = 0 \]

\[ y' = \frac{3y - 6x^5}{6y^5 - 3x} = \frac{y - 2x^5}{2y^5 - x} \]

At $x = 1$ and $y = 0$:

\[ y' = \frac{-6}{-3} = 2 \]
Solution, contd:

For part (b) you want to determine the second derivative to deduce concavity at the given point.

\[(30x^4 - 3y') + (6y^5 - 3x)y'' + y'(30y'y' - 3) = 0\]

(There should be six terms including this one)

at \(x = 1\)
\[y = 0\]
\[y' = 2\]

\[30 - 3y'' - 6 = 0\]
\[18 - 3y'' = 0\]
\[3y'' = 18\]
\[y'' = 6\]

Concave up.