# Chain Rule

## Related Rates

## Implicit differentiation

UBC Math 102

| Teams | points |
| --- | --- |
| Yellow Team | 11 |
| Orange team | 3 |
| Blue Team | 19 |
| Green team | 10 |

| Team size | Teams | points |
| --- | --- | --- |
| 2 | Yellow Team | 11 |
| 1 | Orange team | 3 |
| 3 | Blue Team | 19 |
| 2 | Green team | 10 |

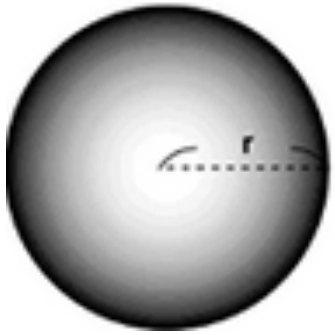
| Team size | Teams | points/person |
| --- | --- | --- |
| 2 | Yellow Team | 5.5 |
| 1 | Orange team | 3 |
| 3 | Blue Team | 6.3 |
| 2 | Green team | 5 |

# Growing sphere:

- Radius increases with time
- => volume also increases with time
- => surface area also increases with time

- Information about one of these can be used to characterize all the other ones!

# Chain Rule




$$V = \frac{4}{3}\pi r^3$$

Time t → r(t) → V(r) →

$$\boxed{\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}}$$

$$V = (4/3)\pi r^3$$

# (1) Growing sphere

$$S = 4\pi r^2$$

If the radius of a sphere increases at a constant rate then the volume of that sphere

(A) Increases at a rate proportional to its surface area.

(B) Increases at a rate proportional to its radius.

(C) Increases at a constant rate.

(D) Increases at a rate proportional to its volume.

(E) Stays constant.

# Growing sphere

Given: $V = \frac{4}{3}\pi r^3$

Remark: $S = 4\pi r^2$
(we'll use this later on)

$$\frac{dr}{dt} = \alpha$$

$$V(r(t)) = \frac{4}{3}\pi \left[r(t)\right]^3$$

· We think of both V and r as time-dependent quantities

By Chain Rule

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

# Growing sphere

By Chain Rule

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

$$= \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$$

$$= \underbrace{4\pi r^2}_{\text{Hey, this looks familiar! it is the surface area!}} \cdot \alpha$$

Remark: $S = 4\pi r^2$
(we'll use this later on)

# Growing sphere

If the radius of a sphere increases at a constant rate then the volume of that sphere satisfies

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3[r(t)]^2 \frac{dr}{dt} = 4\pi r^2 \alpha = \alpha S$$

so the volume of the sphere increases at a rate proportional to its surface area.

# (2) Growing sphere:

$$V = (4/3)\pi r^3$$

$$S = 4\pi r^2$$

Suppose the volume of the sphere is increasing at a constant rate. Then

(A) the radius increases rapidly at first and then slowly.

(B) the radius increases at a constant rate.

(C) the radius increases slowly at first and then much faster.

(D) the radius increases at a rate proportional to the surface area.

(E) none of the above.

- Given: $\frac{dV}{dt} = \beta = \text{constant} > 0$

- But from before: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

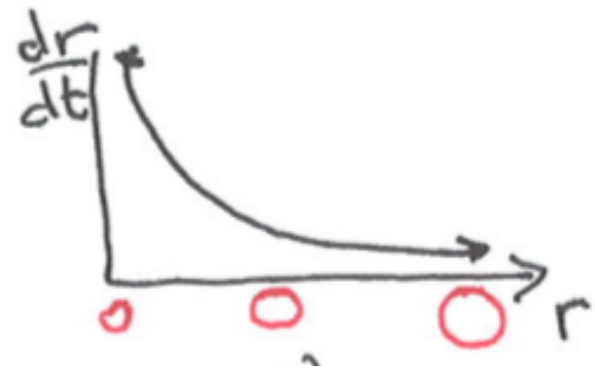
- So: $\beta = 4\pi r^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\beta}{4\pi r^2}$$

(inversely proportional to surface area!)

# Growing sphere II

$$\frac{dr}{dt} = \frac{\beta}{4\pi r^2}$$




-  Small sphere: r changes fast
-  Large sphere: r changes slowly

$$V = (4/3)\pi r^3$$

# Growing sphere

$$S = 4\pi r^2$$

Suppose the volume of the sphere is increasing at a constant rate. Then

$$dV/dt = \beta = \text{constant.}$$

$$\frac{dV}{dt} = \beta = \frac{4}{3}\pi \cdot 3[r(t)]^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$\frac{dr}{dt} = \frac{\beta}{4\pi r^2}$$

# (3) Changing distance:

In the figure shown here, there are two people walking away or towards the street corner. The distances of the individuals from the corner at time $t$ are $x(t)$ and $y(t)$. The distance $L$ is then

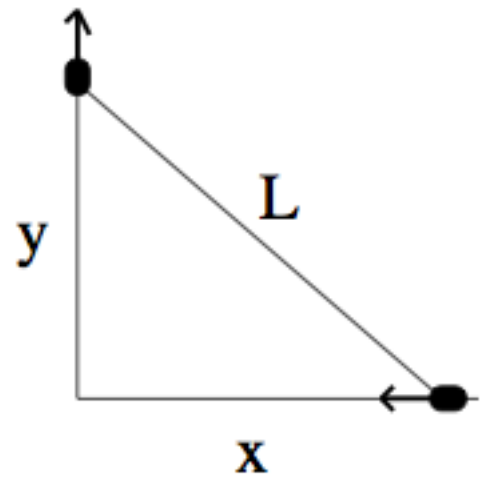
(A) $L = x + y$

(B) $L = y/x$

(C) $L = x/y$

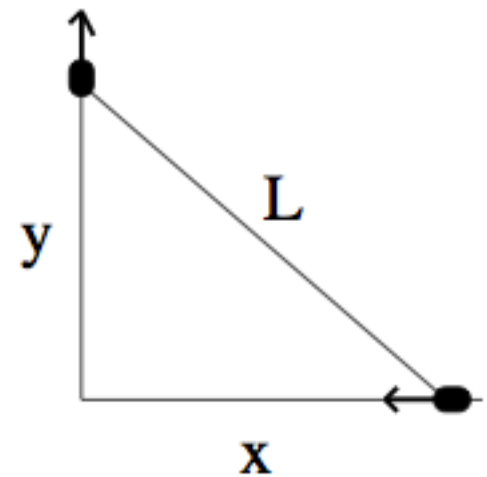
(D) $L = \sqrt{x+y}$

(E) $L = \sqrt{x^2 + y^2}$



# (4) Changing distance:

If in this diagram, one person walks towards the corner at the rate 1m/s and the other walks away at rate 2 m/s, then




(A) $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2,$ (B) $\frac{dx}{dt} = -1, \frac{dy}{dt} = 2,$

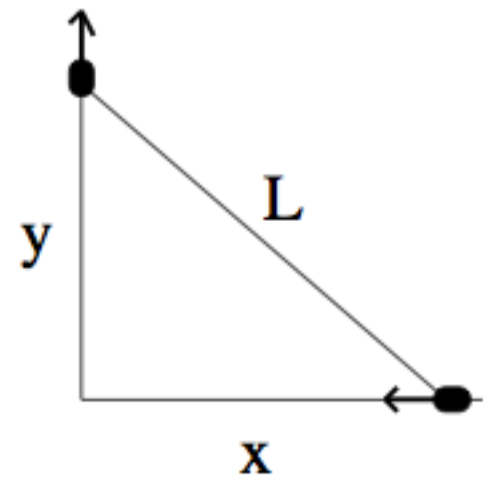
(C) $\frac{dx}{dt} = 1, \frac{dy}{dt} = -2,$

(D) $\frac{dx}{dt} = 2, \frac{dy}{dt} = 1,$ (E) $\frac{dx}{dt} = 2, \frac{dy}{dt} = -1,$

# (5) Changing distance:

If one person walks towards the corner at the rate 1m/s and the other walks away at rate 2 m/s, then how fast (in m/s) is $L$ changing at the instant when $x = y = 10$m?

(A) 2, (B) $\sqrt{2}$, (C) $\frac{1}{\sqrt{2}}$, (D) $\frac{3}{\sqrt{2}}$, (E) 30,



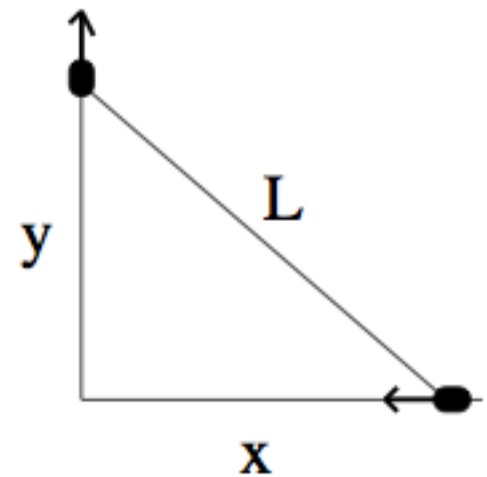
# Changing distance

- The variables are x(t), y(t), L(t)

$$L^2 = x^2 + y^2$$

this just to remind us that all these are changing with time

$$[L(t)]^2 = [x(t)]^2 + [y(t)]^2$$



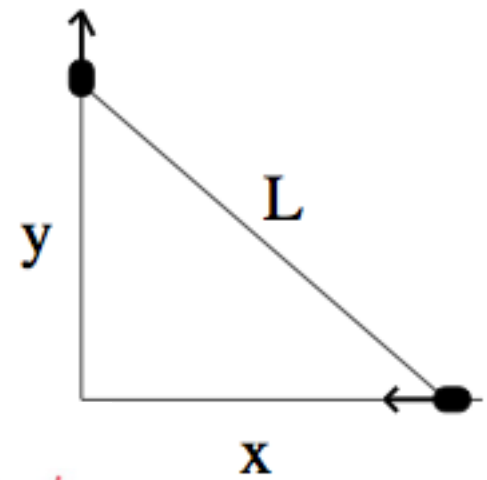

- Given:

$$\frac{dx}{dt} = -1 \quad , \quad \frac{dy}{dt} = 2$$

# Changing distance

$$[L(t)]^2 = [x(t)]^2 + [y(t)]^2$$

Chain rule:




$$\frac{d}{dt}[L(t)]^2 = \frac{d}{dt}\left[[x(t)]^2 + [y(t)]^2\right]$$

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

cancel 2
divide both sides
by L

$$\frac{dL}{dt} = \frac{1}{L}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

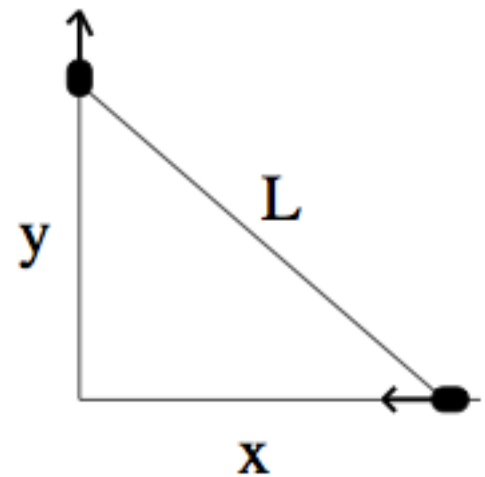
# Changing distance

$$\frac{dL}{dt} = \frac{1}{L}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

- Now use $dx/dt = -1, dy/dt = 2,$

$$x = y = 10$$

$$\left\{\begin{array}{l} x = 10 \\ y = 10 \\ L = \sqrt{10^2+10^2} \\ \quad = 10\sqrt{2} \end{array}\right.$$




$$\frac{dL}{dt} = \frac{1}{\sqrt{2}\cdot 10}\left[10\cdot -1 + 10\cdot 2\right]$$

$$= \frac{1}{\sqrt{2}\cdot \cancel{10}}\,\cancel{10}\,(-1+2) = \frac{1}{\sqrt{2}}$$

- The distance between the two people is increasing at the rate

$$\frac{dL}{dt} = \frac{1}{\sqrt{2}} \quad \text{m/s}$$

# Take-home message:

- Draw and label diagram
- Write down the relationships needed
- Determine which quantities are changing, which are constant.
- Differentiate (carefully), and remember to use the chain rule where needed. (This gives you a general equation that holds for all time)
- Plug in numbers at the end. (This determines a rate of change at one instant)

# Chain Rule

## Implicit differentiation

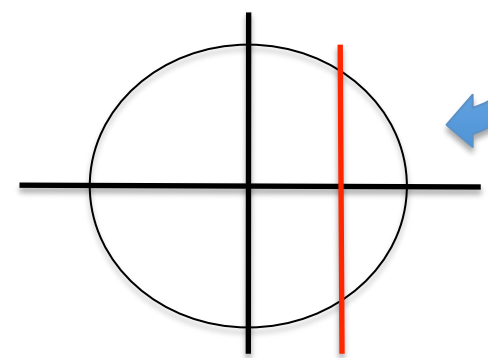
# Implicit differentiation

- Not all curves can be described by 1 function
- Example: a circle – need 2 functions

$$x^2 + y^2 = r^2$$

$$\begin{cases} y = +\sqrt{r^2 - x^2} & \text{top} \\ y = -\sqrt{r^2 - x^2} & \text{bottom} \end{cases}$$

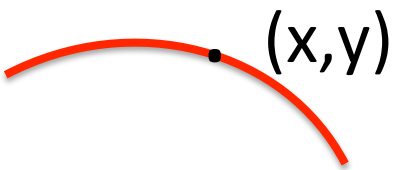
circle does not satisfy the "vertical line" property
∴ cannot be a function such as $y = f(x)$.



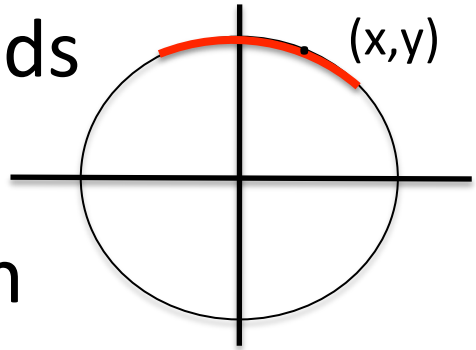
# Implicit differentiation

- However, x and y are linked by the relationship

$$x^2 + y^2 = r^2$$

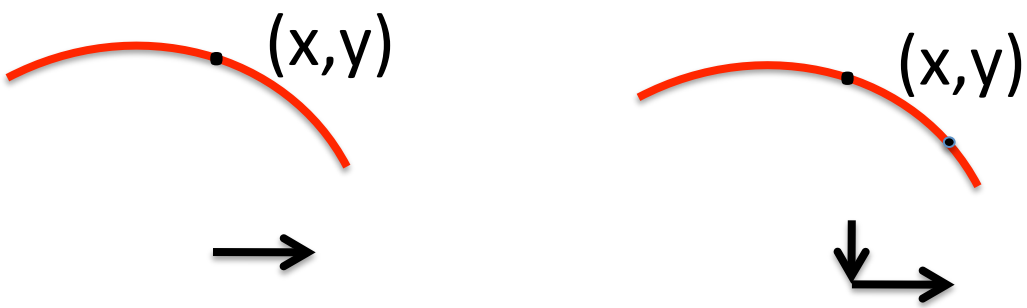



- "Locally" (close to some point) this relationship implies that y depends (uniquely) on x
- We call this an "implicit" function



# Implicit function

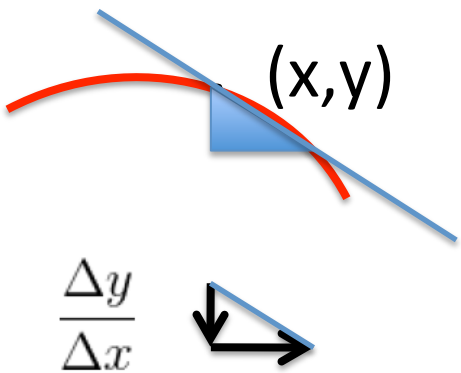
- Locally, if x changes, then y also changes:.

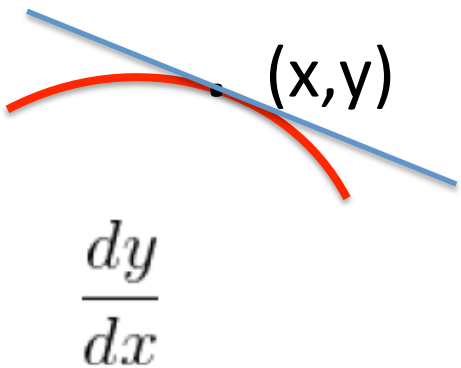



- Locally y depends on x.
- We sometimes write y(x) to denote that.

# Slope

- We can ask what is the slope of the curve at the point (x,y).




Slope of secant line line

Slope of tangent

# We can find $\dfrac{dy}{dx}$ using the chain rule

- Remember that we consider y as a variable that depends on x

$$x^2 + y^2 = r^2$$

$$x^2 + [y(x)]^2 = r^2$$

$$\frac{d}{dx}\left[x^2 + [y(x)]^2\right] = \frac{d}{dx} r^2 = 0$$

r is constant!

so $\frac{dr^2}{dx} = 0$

# Implicit differentiation (chain rule)

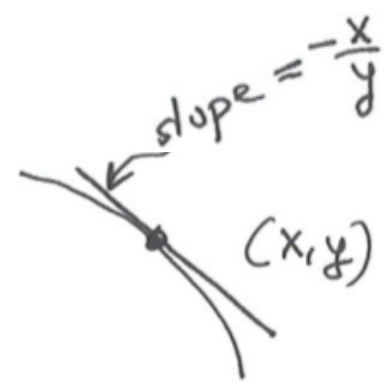
$$\frac{d}{dx}\left[x^2 + [y(x)]^2\right] = 0$$

- Chain rule applied at this step!!

$$2x + \frac{d[y^2]}{dy}\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}}$$



# Example 9.5:

Find the slope of the tangent line to the point $x = 1/2$ in the first quadrant on a circle of radius 1 and center at the origin.

# (6) Example 9.5:

Find the slope of the tangent line to the point $x = 1/2$ in the first quadrant on a circle of radius 1 and center at the origin.

The slope is:

(A) $\frac{\sqrt{3}}{3}$ (B) $-\frac{\sqrt{3}}{3}$ (C) $-1$ (D) $-\frac{1}{2\sqrt{2}}$ (E) $-\frac{\sqrt{2}}{2}$

# Example 9.5: solution

Find the slope of the tangent line to the point x = 1/2 in the first quadrant on a circle of radius 1 and center at the origin.

- At x=1/2 get y= $\sqrt{1-(1/4)} = \sqrt{3/4} = \sqrt{3}/2$

$$\frac{dy}{dx} = -\frac{1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}.$$

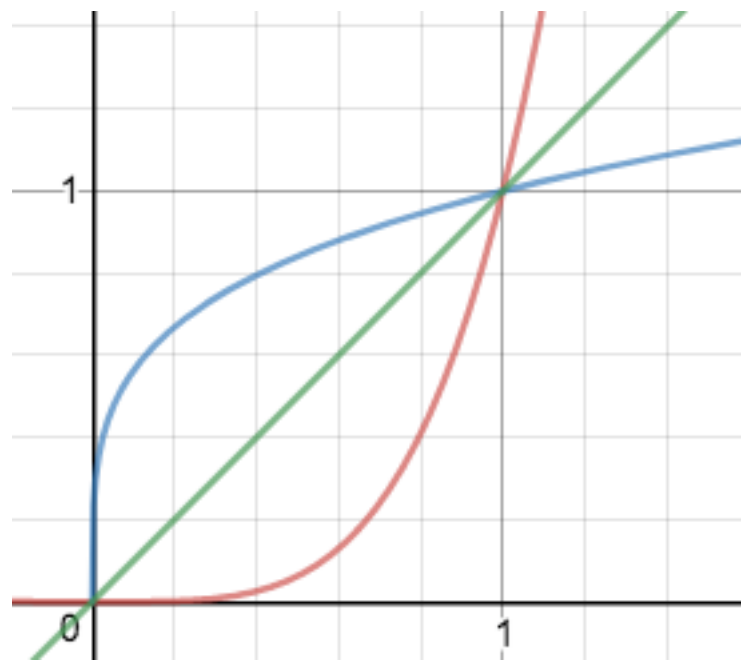
# Inverse functions

(An example of why implicit differentiation is so helpful)

$f(x) = x^n$

$g(x) = x^{\frac{1}{n}}$

inverse functions



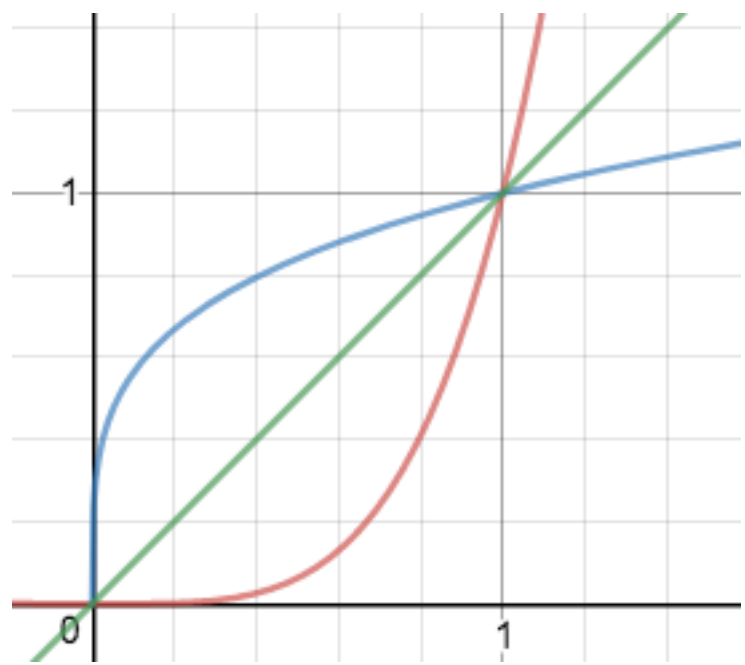
# Use implicit differentiation to verify the power rule for fractional power

Consider the function $y = g(x) = x^{1/n}$

Find its derivative by implicit differentiation after rewriting it as $y^n = x$

$f(x) = x^n$

$g(x) = x^{\frac{1}{n}}$



# Solution:

$$y = g(x) = x^{1/n}$$

We rewrite the relationship in terms of the inverse function $y^n = x$

We consider y as a variable that depends on x and we differentiate both sides

$$[y(x)]^n = x \quad \Rightarrow \quad \frac{d[y(x)]^n}{dx} = 1$$

Now use the chain rule: $$ny^{n-1}\frac{dy}{dx} = 1$$

Finally, isolate dy/dx: $$\frac{dy}{dx} = \frac{1}{ny^{n-1}}$$

# Soln, contd:

- Simplify where possible, using the relationship of y and x:

$$\frac{dy}{dx} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{nx^{(n-1)/n}}$$

$$= \frac{1}{nx^{1-\frac{1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

$$\boxed{g(x) = x^{\frac{1}{n}} \quad \Rightarrow \quad g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}}$$

# Comment

- Implicit differentiation is very useful for finding derivative of inverse functions

- We will see many examples where this is so.

- The present example: used known derivative of the function $y = x^n$ (power rule) to find the derivative of $y = x^{1/n}$

# Example:

Find the slope of the tangent line at the point (1,1) on the curve

$$x^2 + xy + y^2 = 3$$

- Hint: a product rule is needed!

# (7) Example:

Find the slope of the tangent line at the point (1,1) on the curve

$$x^2 + xy + y^2 = 3$$

The slope is

(A) 1 (B) -2/3 (C) -3 (D) -1 (E) 3

# Example:

Find the slope of the tangent line at the point (1,1) on the curve

Product rule

$$x^2 + \boxed{xy} + y^2 = 3$$

$$2x + \overbrace{x\frac{dy}{dx} + 1\cdot y} + 2y\frac{dy}{dx} = \frac{d3}{dx} = 0$$

$$2x + y + \frac{dy}{dx}(x+2y) = 0$$

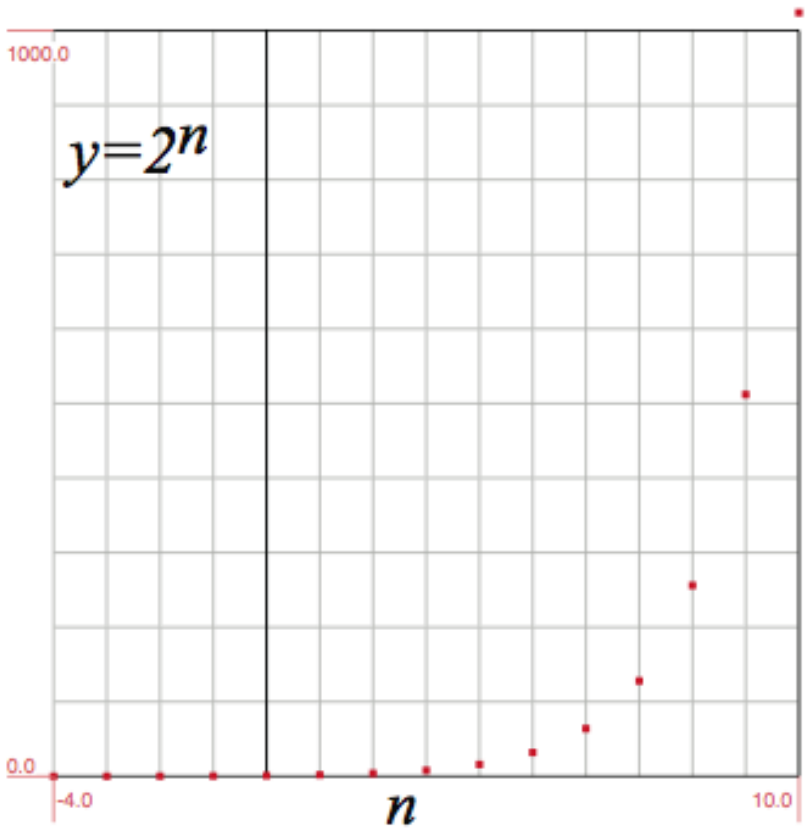
$$\frac{dy}{dx} = \frac{-(2x+y)}{(x+2y)}$$

$$\frac{dy}{dx} = -\frac{3}{3} = -1$$

# Exponential functions and exponential growth

UBC Math 102

# The function $2^n$




| $n$ | $2^n$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

# (1) Which of the following approximations is reasonable?

(A) $2^{10} \approx 100$

(B) $2^{10} \approx 500$

(C) $2^{10} \approx 1000$

(D) $2^{10} \approx 5000$

(E) $2^{10} \approx 10000$

# (2) Andromeda strain

*"The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."*

Michael Crichton (1969) The Andromeda Strain, Dell, N.Y. p247

https://www.youtube.com/watch?v=gEwzDydciWc

This statement is

(A) TRUE (B) FALSE

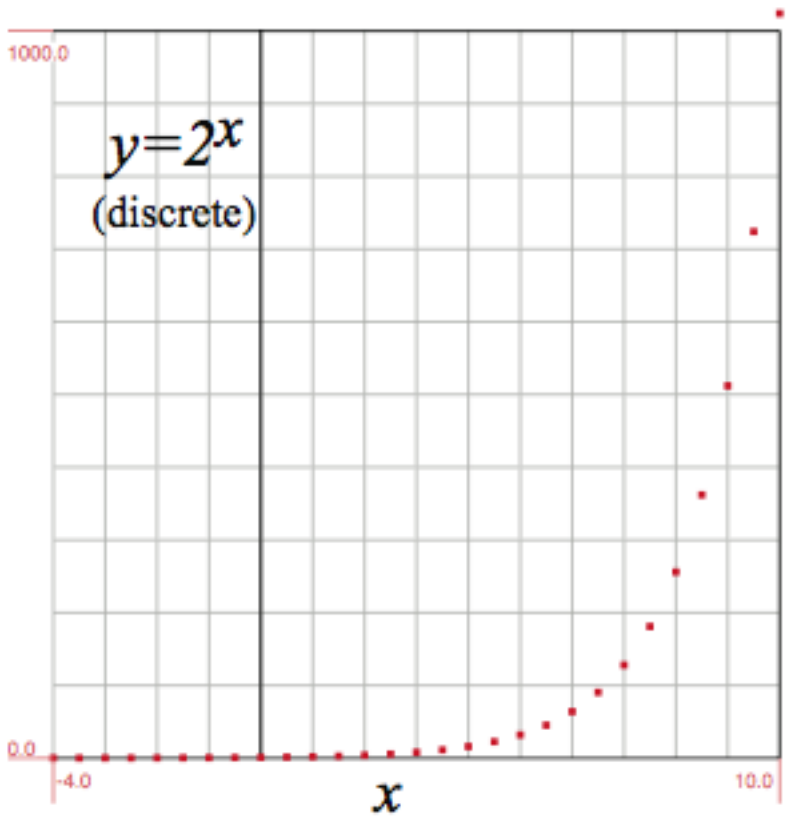
# Exponential growth application

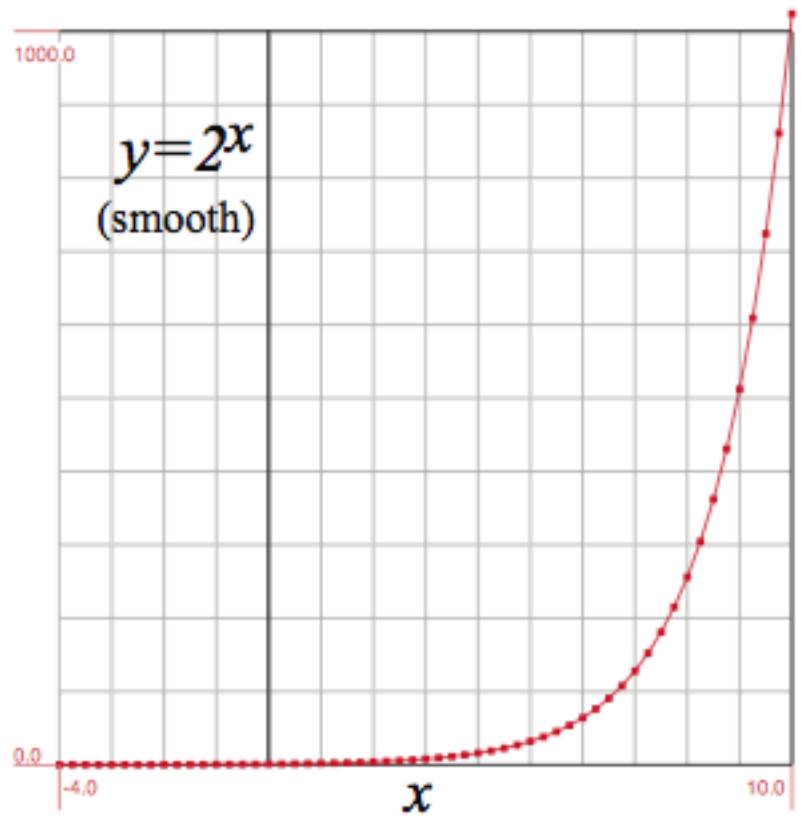
POLYMERASE CHAIN REACTION

- A way of amplifying DNA for research, genetics, crime-scene forensics

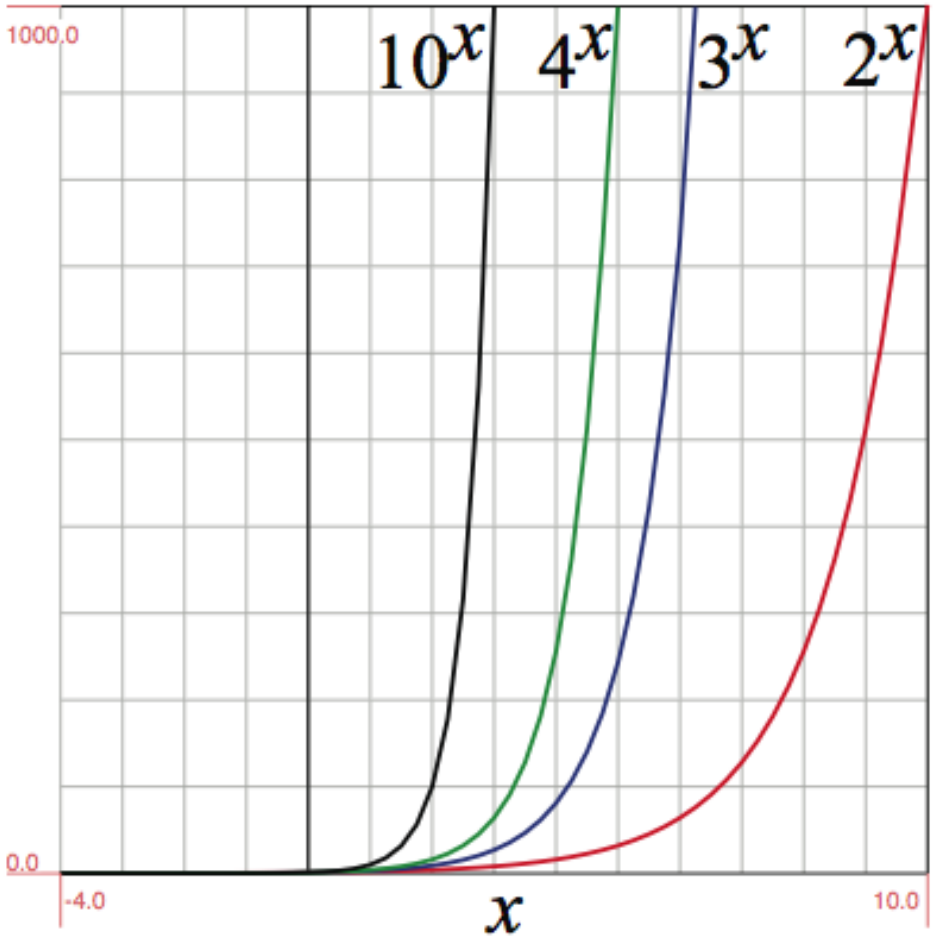
- https://www.youtube.com/watch?v=HMC7c2T8fVk
- See also the PCR Song:
- https://www.youtube.com/watch?v=x5yPkxCLad

# A smooth function $2^x$



# Other bases: $y=a^x$



# What is the derivative of $y=a^x$ ?

Use the definition of the derivative to calculate dy/dx.

# (3) What is the derivative of $y=a^x$ ?

(A) $a^x$

(B) $xa^{x-1}$

(C) $Cxa^{x-1}$ where C is some constant

(D) $ax^{a-1}$

(E) $Ca^x$ where C is some constant

# (3) What is the derivative of $y=a^x$ ?

Use the definition of the derivative to calculate dy/dx.

$$\begin{aligned}
\frac{da^x}{dx} &= \lim_{h\to 0} \frac{\left(a^{x+h} - a^x\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(a^x a^h - a^x\right)}{h} \\
&= \lim_{h\to 0} a^x \frac{\left(a^h - 1\right)}{h} \\
&= a^x \boxed{\left[\lim_{h\to 0} \frac{a^h - 1}{h}\right]} \\
&= C a^x
\end{aligned}$$

This thing is a constant

**The derivative of an exponential function $a^x$ is of the form $C_a a^x$ where $C_a$ is a constant that depends only on the base $a$.**

# (3) What is the derivative of $y=a^x$ ?

Use the definition of the derivative to calculate dy/dx.

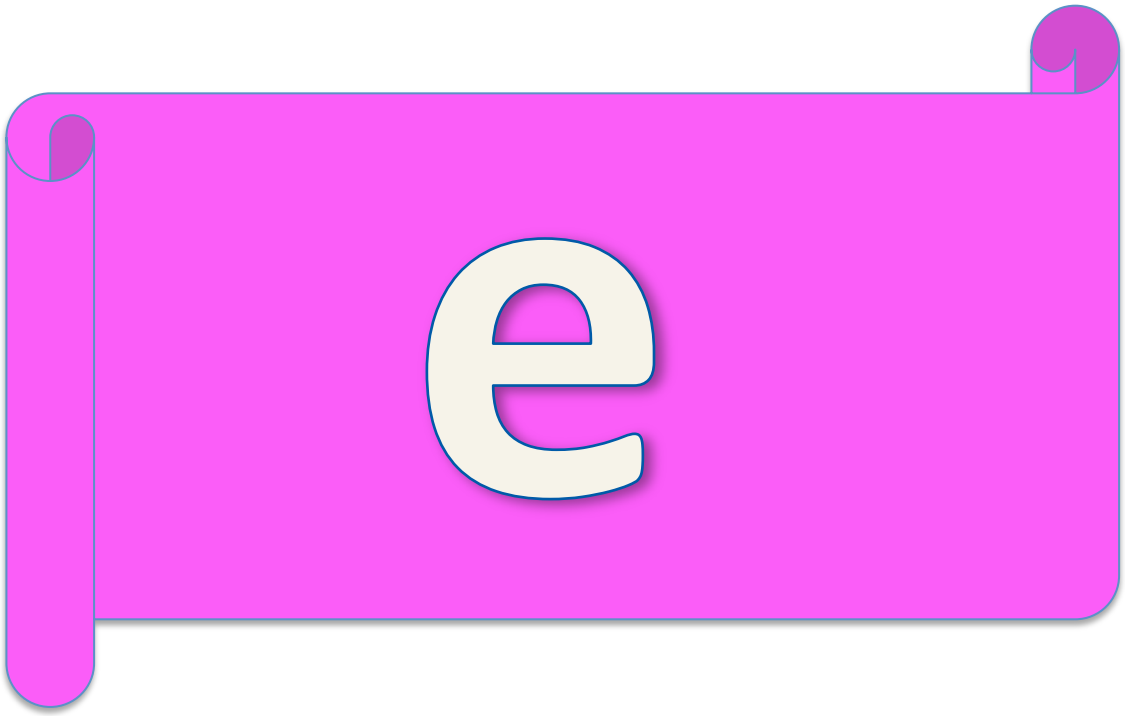
$$\begin{aligned}
\frac{da^x}{dx} &= \lim_{h\to 0} \frac{\left(a^{x+h} - a^x\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(a^x a^h - a^x\right)}{h} \\
&= \lim_{h\to 0} a^x \frac{\left(a^h - 1\right)}{h} \\
&= a^x \boxed{\left[\lim_{h\to 0} \frac{a^h - 1}{h}\right]} \\
&= C a^x
\end{aligned}$$

IDEA: pick a convenient base such that this constant is C= 1

This thing is a constant

**The derivative of an exponential function $a^x$ is of the form $C_a a^x$ where $C_a$ is a constant that depends only on the base $a$.**

# Introducing a special base



# What's so special?

The derivative of $e^x$ is $e^x$

# Comparison

POWER Function

$y=f(x)=x^n$

$f'(x) = n\, x^{n-1}$

$f''(x) = (\ )\, x^{n-2}$

…

$f^{(n)}(x) = n(n-1)\ldots 1$

$f^{(n+1)}(x) = 0$

EXPONENTIAL Function

$y=f(x)=e^x$

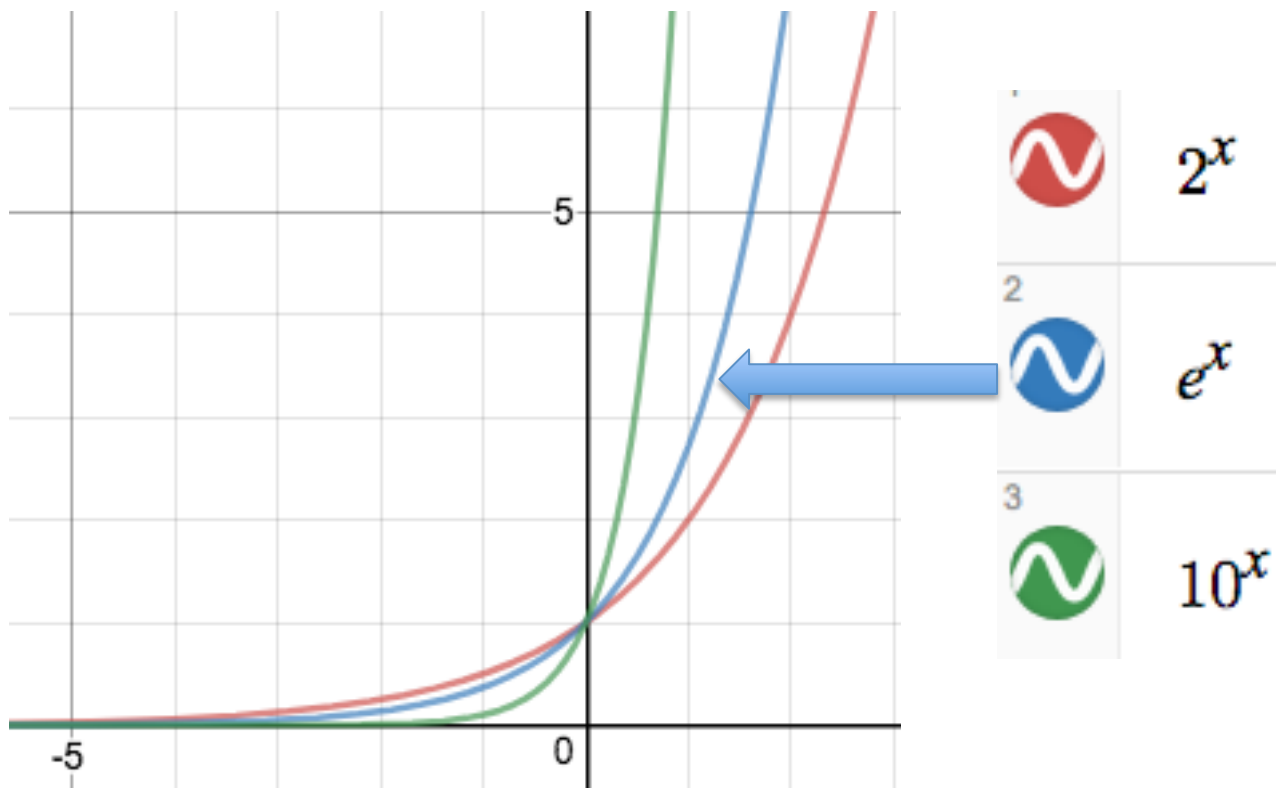
$f'(x)=e^x$

$f''(x)=e^x$

…

# How does $e^x$ compare with other bases?

Its graph looks a lot like that of other common bases!



# So why should we care about it?

- We use this exponential function mostly because its derivative is more convenient than the derivatives of the other exponential functions.

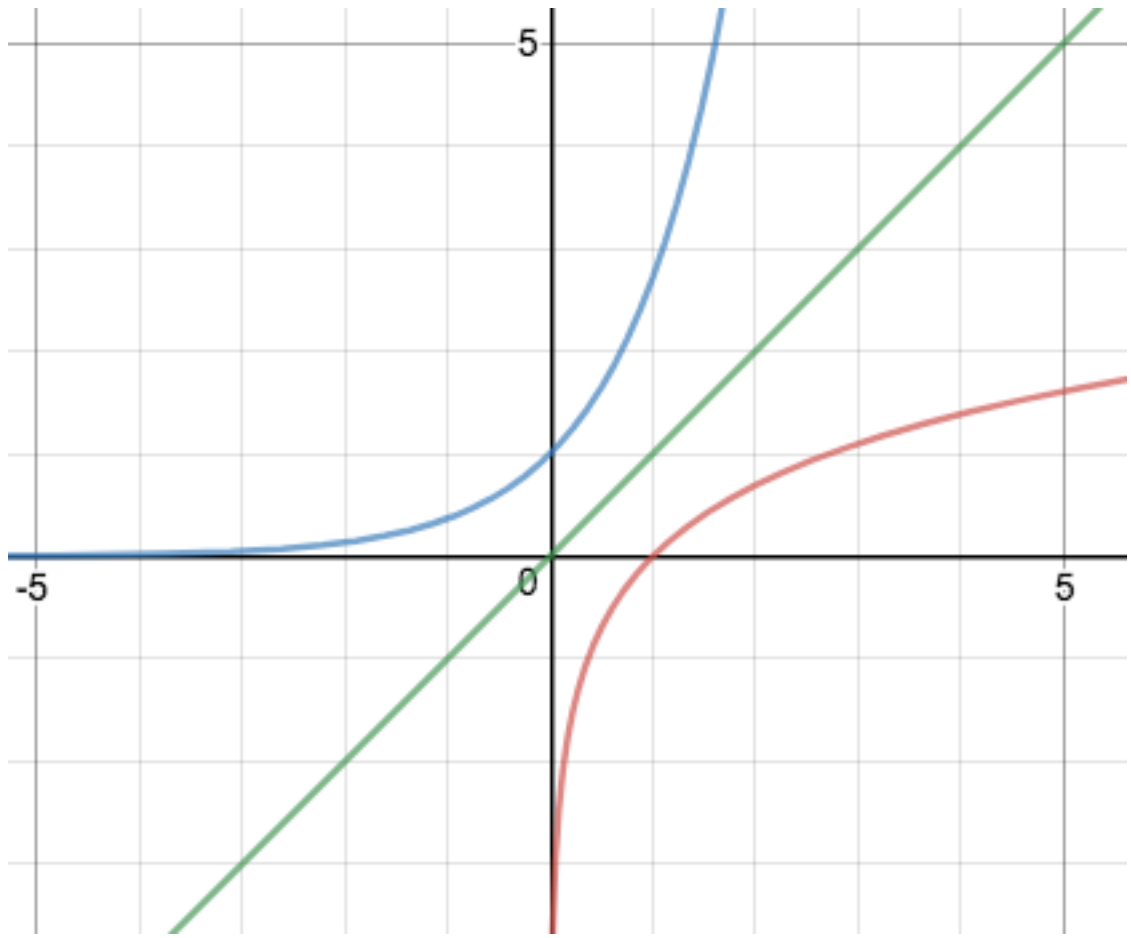
# Ln(x) is the inverse function for $e^x$

# The natural logarithm is an inverse function for $e^x$

1 $\ln(x)$

2 $e^x$

3 $y = x$



# Derivative of $\ln(x)$ by implicit differentiation

- Restate the relationship in the form

$$y = \ln(x) \quad \Rightarrow \quad e^y = x$$

- Now use implicit differentiation to find dy/dx

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$$\frac{d}{dx} e^{y(x)} = \frac{d}{dx} x$$

$$\frac{de^y}{dy}\frac{dy}{dx} = 1 \quad \Rightarrow \quad e^y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

- Result:

$$\boxed{\frac{d\ln(x)}{dx} = \frac{1}{x}.}$$

# Answers

- ChainRule & Implicit Diff
- 1 A
- 2 A
- 3 E
- 4 B
- 5 C
- 6 B
- 7 D

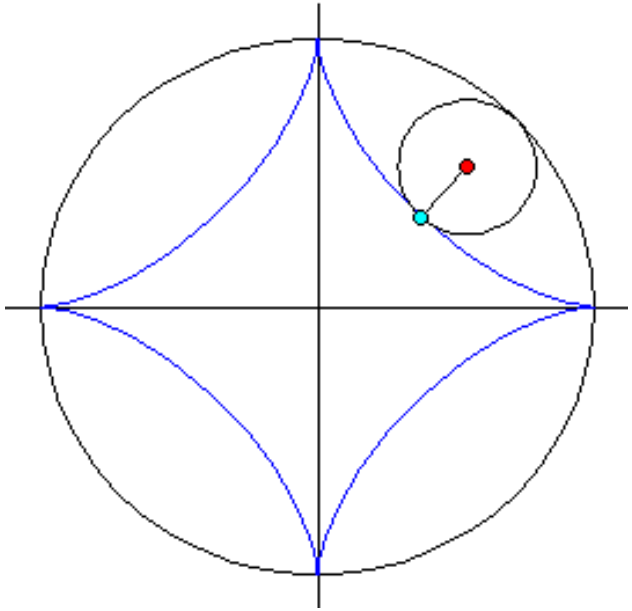
- Exponentials
- 1 C
- 2 B
- 3 E

# QUESTIONS TO TEST YOUR SKILLS

# Implicit differentiation example: Astroid

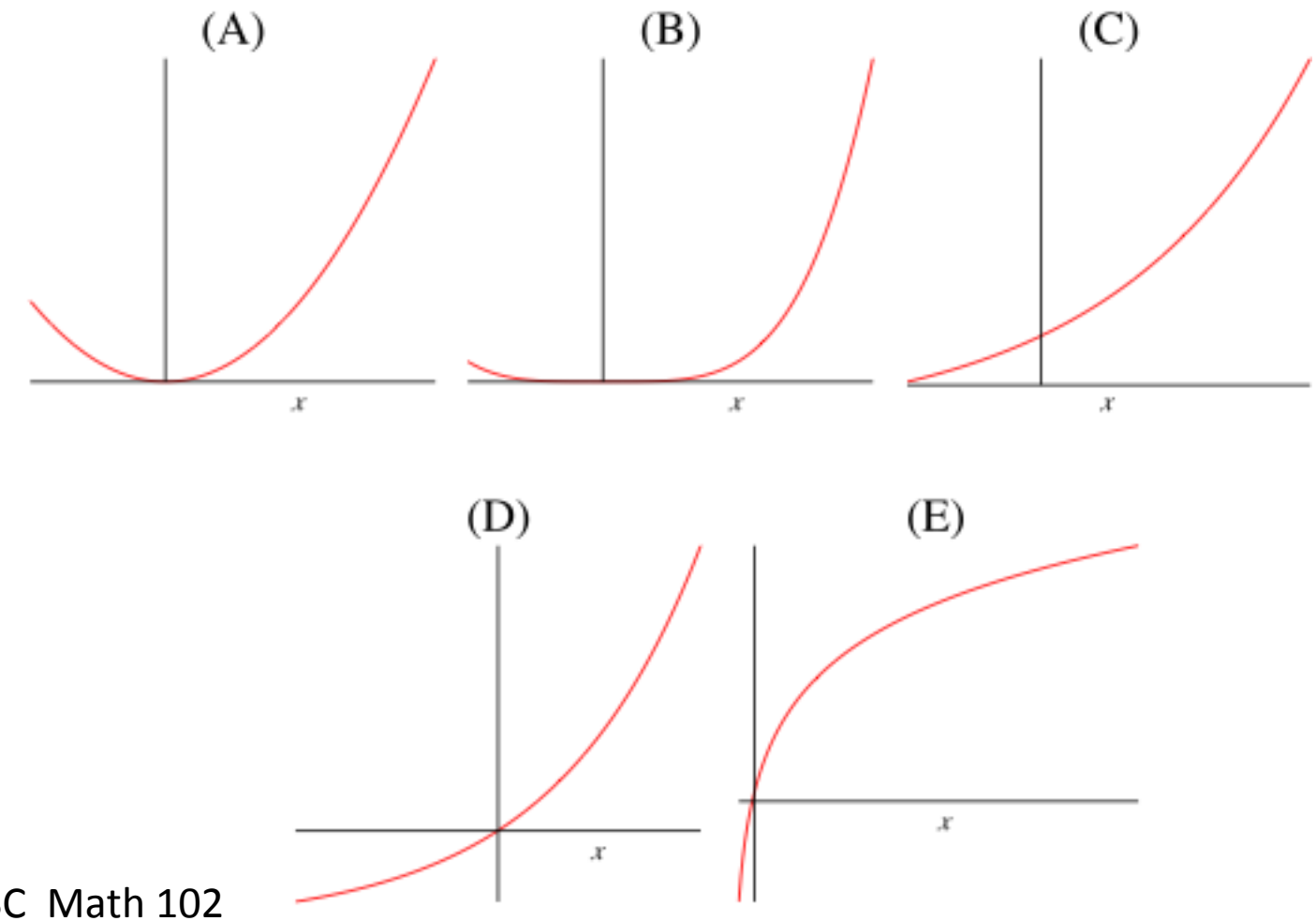
The curve $x^{2/3} + y^{2/3} = 2^{2/3}$ has the shape of an astroid. Find the slope of the tangent line to a point on the astroid.

(See Example 9.8)



https://www.youtube.com/watch?v=CeQDxV1vGqM&feature=youtu.be

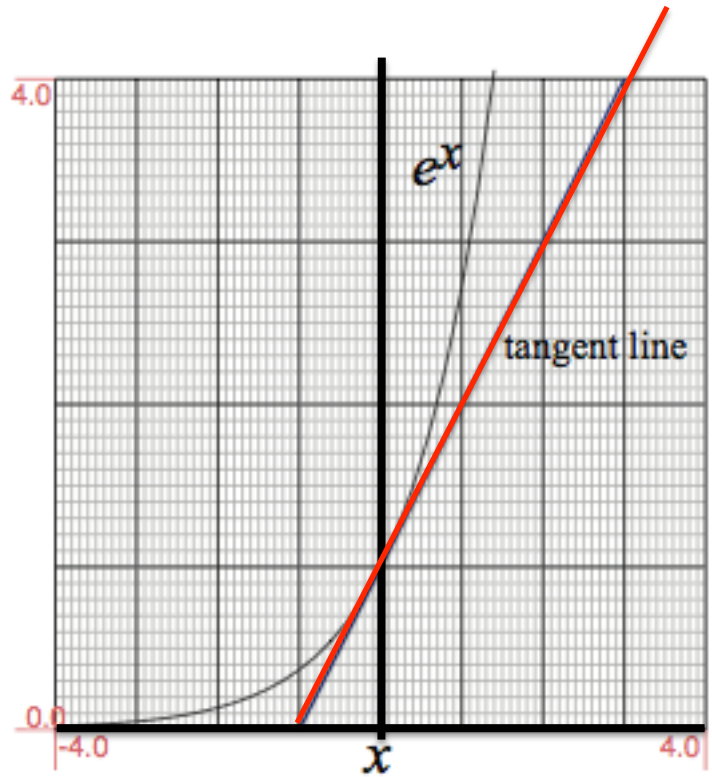
# (10) The graph of $e^x$ looks like which of these?



# Find the equation of this tangent line

Tangent line to the curve

$y=e^x$ at $x=0$.



# Exponential: Exam question

Note: Part (a) uses logarithm in final step.. See next lecture

16.14. **Fish generations:** In Fish River, the number of salmon (in thousands), $x$, in a given year is linked to the number of salmon (in thousands), $y$, in the following year by the function

$$y = Axe^{-bx}$$

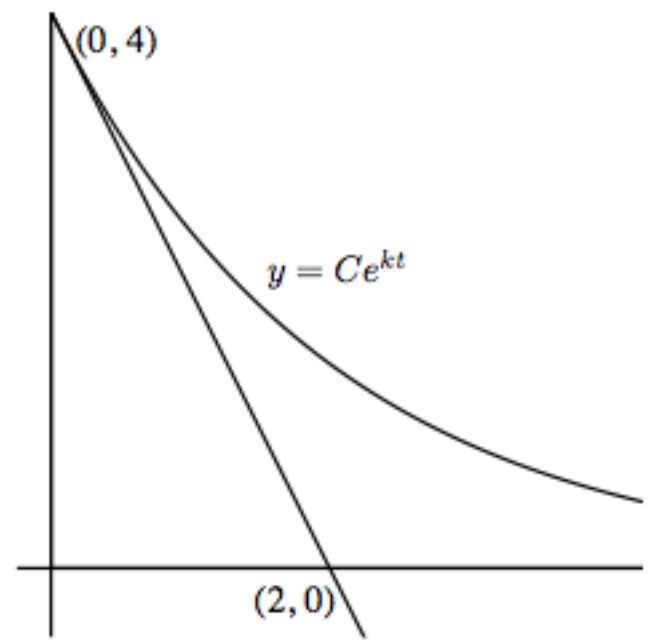
where $A, b > 0$ are constants.

(a) For what number of salmon is there no change in the number from one year to the next?

(b) Find the number of salmon that would yield the largest number of salmon in the following year.

# Practice: conceptual midterm question

Here is the graph of $y = C\,e^{kt}$ for some constants C, k, and a tangent line. Use data from the graph to determine C and k.



# Practice midterm question

Sketch the graph of the function $y = x^2 e^{-x}$

# Practice Exam question

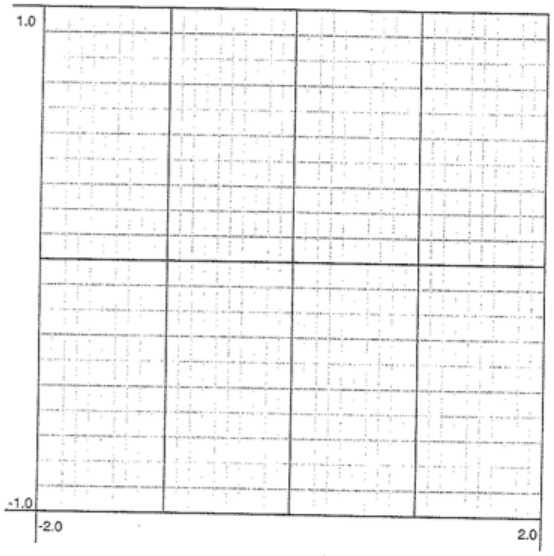


[10] **4.** Consider the function

$$y = f(x) = (1 - x^2)e^{-x^2}$$

(a) Find all critical points. (Note: the second derivative is not required in this problem.)

(b) Sketch the graph of this function on the grid provided. (Note that the $x$ axis scale is $-2 < x < 2$ and the $y$ axis scale is $-1 < y < 1$. Also: it may or may not be helpful to use the following facts: $\sqrt{2} \approx 1.41$, $e \approx 2.72$, $1/e \approx 0.37$ $1/e^2 \approx 0.135$.)



# Practice Exam Question:

2: Consider the function $y = \log_2(x)$. The slope of this function at the point $x = 1$ is

(a) 1 (b) $1/\ln(2)$ (c) $\ln(2)$ (d) $2\ln(2)$ (e) $2/\ln(2)$

# Practice Exam Question

1: Consider the function $y = f(x) = 3e^{-2x} - 5e^{-4x}$

(a) The function has a local maximum at $x = (1/2)\ln(10/3)$

(b) The function has a local minimum at $x = (1/2)\ln(10/3)$

(c) The function has a local maximum at $x = (-1/2)\ln(3/5)$

(d) The function has a local minimum at $x = (1/2)\ln(3/5)$

(e) The function has a local maximum at $x = (-1/2)\ln(3/20)$

# Practice Exam question for Implicit differentiation



Question 4: [10 points] Consider the curve whose equation is

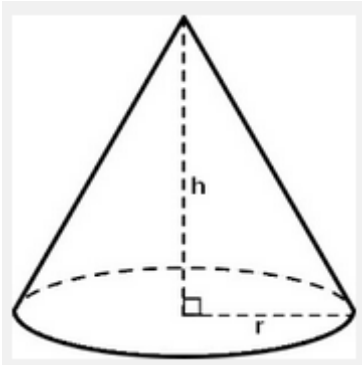
$$x^6 - 3xy + y^6 = 1$$

(a) Find the slope of the tangent line at the point (1, 0) on this curve.

(b) Determine whether the curve is concave up or concave down at the point (1, 0).

# Solutions to previous problems

# Conical pile of sugar




Problem from last time

Sugar is poured at a constant flow rate, 1 $cm^3/s$ to form a conical pile. At what rate is the base area increasing?

$$V = \frac{1}{3}\pi r^2 h$$

Solution

# Solution: first express V in terms of A

$$V = \frac{1}{3}\pi r^2 h \qquad h = Cr$$

$$= \frac{1}{3}\pi C r^3 \qquad A = \pi r^2$$

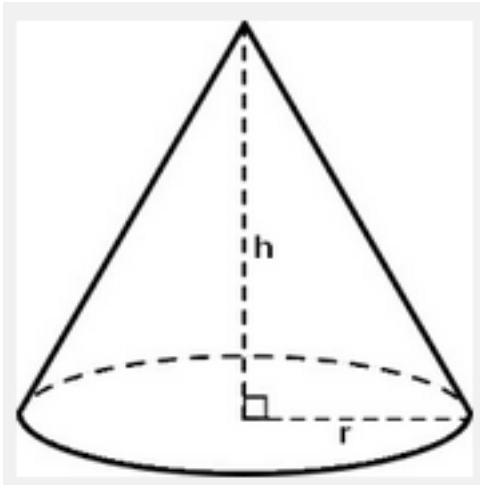
$$\left(\frac{A}{\pi}\right)^{1/2} = r$$

$$V = \frac{1}{3}\pi C \left(\frac{A}{\pi}\right)^{3/2}$$

$$= \frac{1}{3}\pi^{1-3/2} C A^{3/2}$$

$$= \frac{C}{3\sqrt{\pi}} A^{3/2} \quad \leftarrow$$

# Now use Chain Rule




Time t → A(t) → V(r) →

$$\frac{dV}{dt} = \frac{dV}{dA} \cdot \frac{dA}{dt}$$

# Solution, cont'd

$$\frac{dV}{dt} = \frac{C}{3\sqrt{\pi}} \cdot \frac{d}{dt}\left[A^{3/2}\right]$$

$$= \frac{C}{3\sqrt{\pi}} \cdot \frac{d}{dA}\left[A^{3/2}\right] \cdot \frac{dA}{dt}$$

$$= \frac{C}{\cancel{3}\sqrt{\pi}} \cdot \frac{\cancel{3}}{2} A^{1/2} \frac{dA}{dt}$$

$$\underset{\substack{\uparrow \\ 1\,cm^3/s}}{\frac{dV}{dt}} = C\sqrt{\frac{A}{\pi}} \frac{dA}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{C}\sqrt{\frac{\pi}{A}} \quad cm^2/s$$

# Related test problem

3. [**3 pt**] Oil is leaking out of a cargo ship at the rate of 1 $m^3$/hr, forming a circular patch on the surface of the water (an "oil slick"). The radius $r(t)$ of the oil slick increases while its thickness, $\tau = 0.01$ m, is constant. Find the rate of change of the radius, at the moment when $r = 10$ m. Reminder: The volume of a cylinder is $V = \pi r^2 h$.



Solution

# Solution:


$$v(t) = \pi r(t)^2 \tau$$

$$1\,m^3/hr = v'(t) = 2\pi\tau\, r(t) r'(t) \quad 1$$

$$r'(t) = \frac{1}{2\pi\tau\, r(t)} \quad 1$$

$$r'(t_0) = \frac{1}{2\pi \frac{1}{100} 10} = \frac{5}{\pi}\ m/hr \quad 1$$