Midterm information

- What does it cover?
- What should I expect?

See: Midterm Information page on the wiki

What's due and when?

- OSH 4:

- Please note that the deadline is

  8:59 pm, Fri, Oct 20

Do not count on being able to submit later, or to send it by email after the deadline.

Least squares data fitting; Chain Rule

Least squares
(Linear regression)
data fitting cont'd..

From last time.

- We are given some data and want to describe its trend.
- How do we fit the best line through the data?

- This is a practial application of calculus, because *it involves minimization.*

- "*Least* squares fitting" – a useful (and mathematically simple) procedure

Learning goals

- Understand what data fitting means in the simplest (linear least squares) setting.
- Understand the connection to optimization
- Be able to fit a line y=ax +b data points
- Instructions for how to use a spreadsheet to fit a line to data

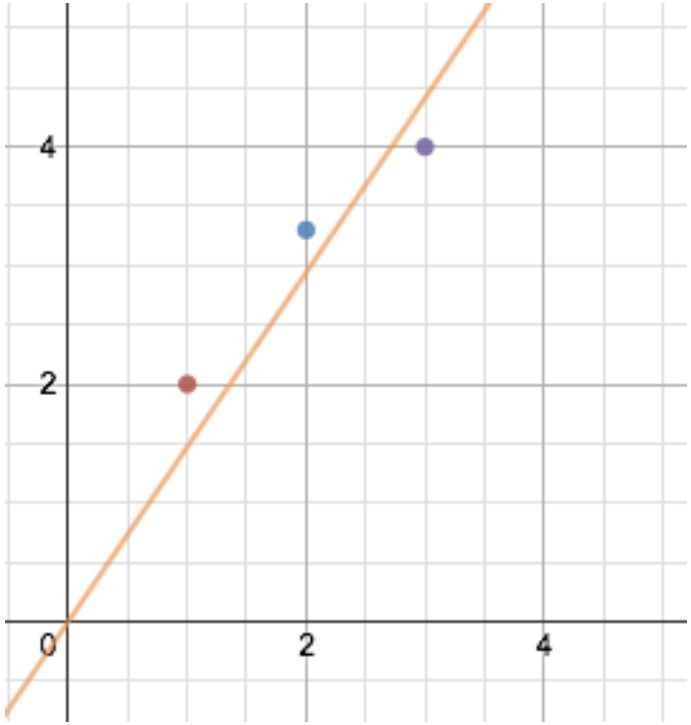
Last time: Example

- Rain fall over three days

Day	1	2	3
Rain (cm)	2	3.3	4

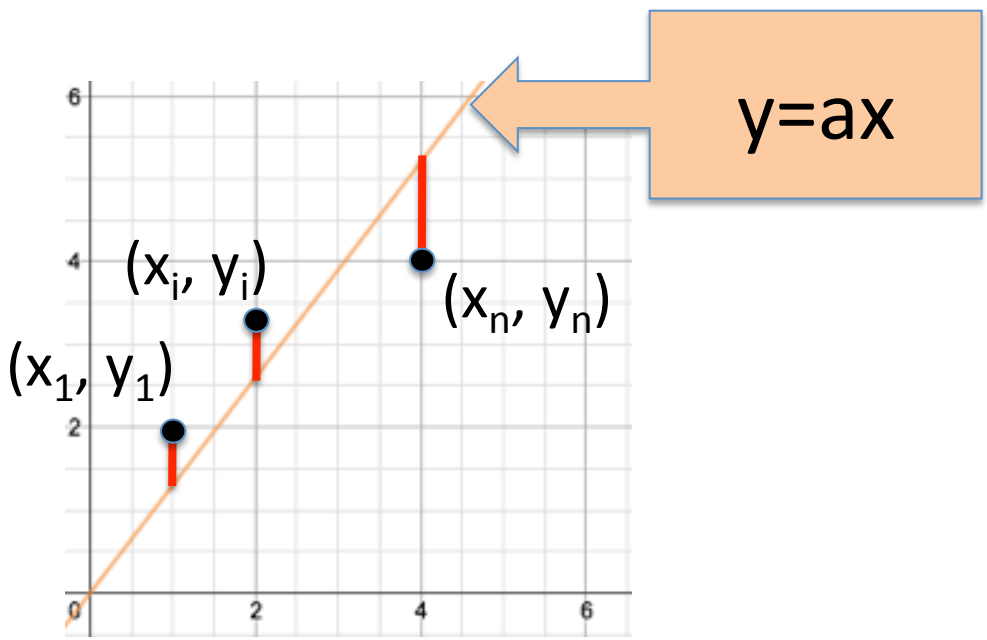
- Find *a* for "best line"

$$y = ax$$



What is “best” line?

- Chose line for which the residuals are as small as possible! (minimize!)

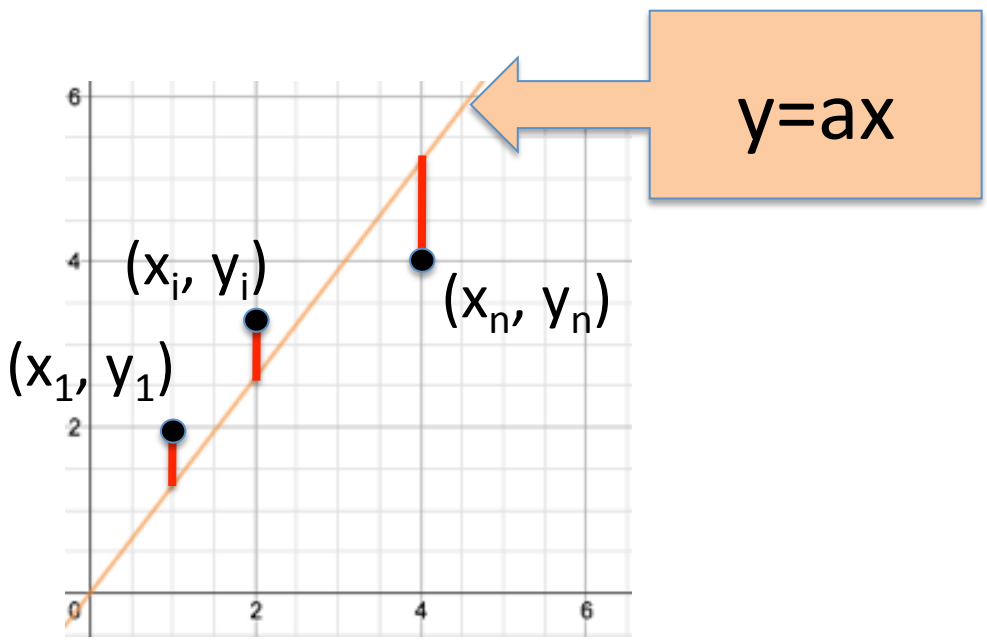


$y = ax$

What is a residual?

Residual = data value - theoretical value

$= y_i - a\,x_i$



"Sum of square residuals"

- SSR= $(y_1-ax_1)^2 + (y_2-ax_2)^2 + \ldots + (y_n-ax_n)^2$
- Short-hand notation:

  SSR(a) = $\sum_{i=1}^{N}(y_i - ax_i)^2$

Minimizing SSR(a) is equivalent to finding the slope of the line for which the deviations of data from the line are smallest overall.

Line y=*a*x; best fit value of *a*:

$$a = \frac{\sum y_i x_i}{\sum x_i^2}$$

- We showed that this was obtained by minimizing SSR
- The optimal value of the slope of the line to fit the data!

Definitions

- A ***model*** is a function used to represent or fit data. For example, some common ones: $f(x)=ax$, $f(x)=ax+b$, $f(x)=Ce^{-kx}$.
- ***Residuals*** are a measure of how far each model value is from the data value: $r_i=y_i-f(x_i)$.
- The ***Sum of Squared Residuals*** (SSR) is a measure of how well the model fits all the data: $SSR = \sum (y_i-f(x_i))^2$.
- Smaller SSR is better.

“Best fit”

- The best fit model is the model with parameter value(s) (*a*, *a* and *b*, etc) that gives the smallest SSR.

(1) Best fit line with intercept $y=ax+b$

- The residuals are

(A) ax_i+b

(B) $(ax_i+b)^2$

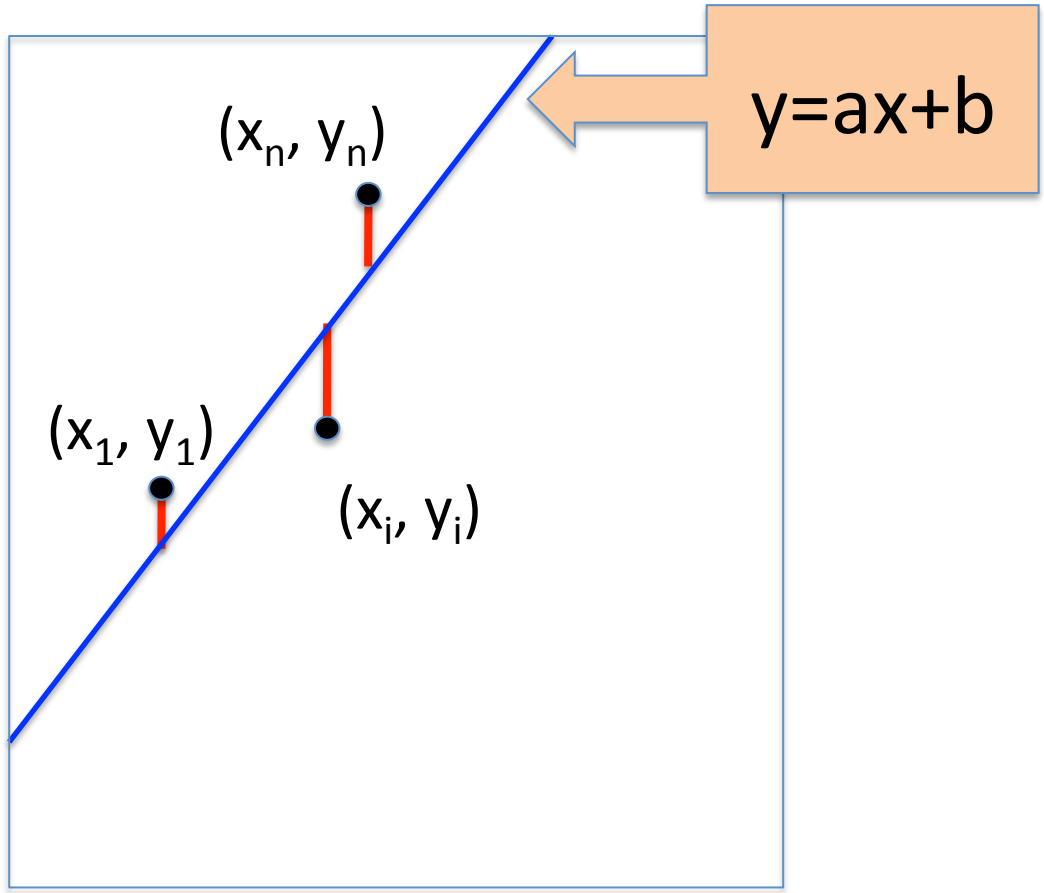
(C) $y_i-(ax_i+b)$

(D) $y_i^2-(ax_i+b)^2$

(E) y_i-ax_i

Residuals for line y=ax+b

- $r_i = y_i - (ax_i + b)$



(2) For best fit line with intercept $y=ax+b$

- The residuals are $y_i-(ax_i+b)$ and we will minimize

(A) $\Sigma\ y_i-(ax_i+b)$

(B) $\Sigma\ |y_i-(ax_i+b)|$

(C) $\Sigma\ (y_i-(ax_i+b))^2$

(D) $\Sigma\ \ y_i^2-(ax_i+b)^2$

(E) None of the above

(3) For best fit line with intercept $y=ax+b$

- We will minimize $SSR=\Sigma\ (y_i-(ax_i+b))^2$

With respect to

(A) a
(B) b
(C) x_i
(D) x_i and y_i
(E) both a and b

(4) How would we do that?

Find value of *a* and b such that

(A)d(SSR)/d*a=0*

(B) d(SSR)/d*b=0*

(C) Both (A) and (B)

(D) none of the above

(E) No clue

Challenge!

- For fun: Calculate values of of *a* and b such that

$$d(SSR)/da=0 \quad \text{AND} \quad d(SSR)/db=0$$

Where $SSR=\Sigma\,(y_i-(ax_i+b))^2$

Best fit line with intercept y=ax+b

- RESULT: (See Supplement on wiki, no need to memorize these!)

$$a = \frac{P_{avg} - \bar{x}\bar{y}}{X^2_{avg} - \bar{x}^2} \qquad b = \bar{y} - a\bar{x}$$

- Where:

$$P_{avg} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$X^2_{avg} = \frac{1}{n}\sum_{i=1}^{n} (x_i^2) \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Best fit line with intercept y=ax+b

- Many spreadsheets (excel, Google-sheets) will compute such lines for you automatically
- Some slides follow with instructions. Or see:

M102 Wiki Spreadsheet help

Assignment7: Problem 14

Using a spreadsheet to fit a trend-line to data:

y is the total volume of all chloroplasts inside a given cell with volume *x*

Wanted: given data, fit $y = ax,$ *to it*

Many spreadsheets will fit data

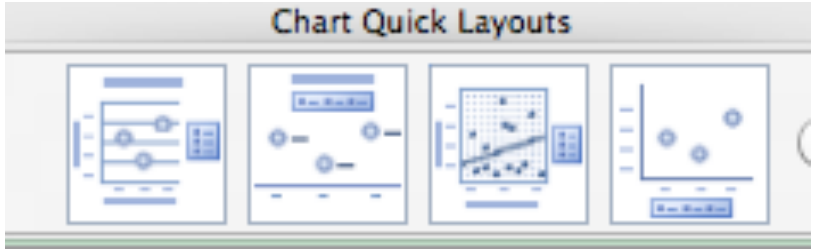
Excel:

Chart 1

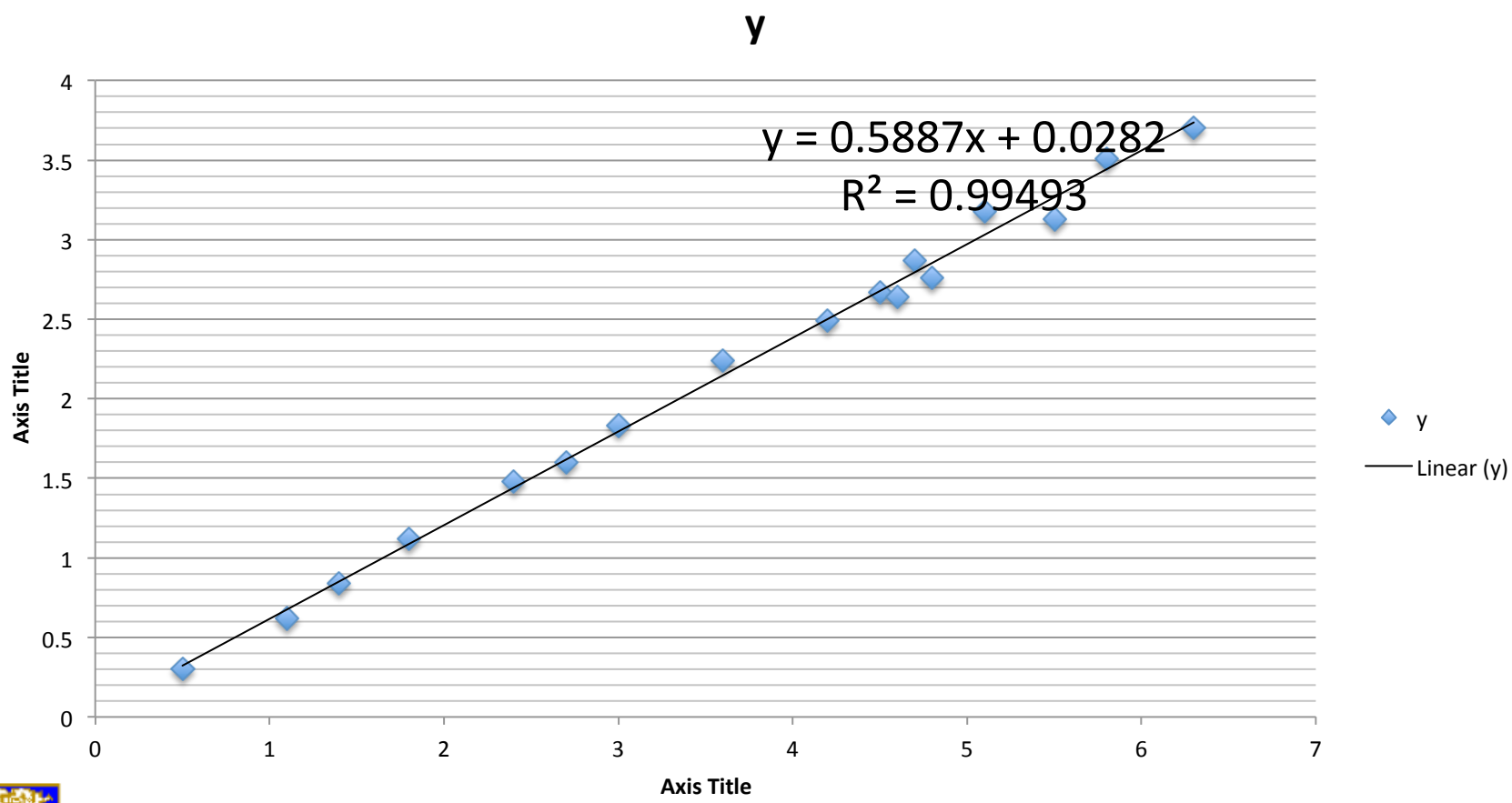
	A	B	C	D	E
1	i	1	2	3	4
2	x	0.5	1.1	1.4	1.8
3	y	0.3	0.62	0.84	1.12
4					

Highlight the rows containing x an y values

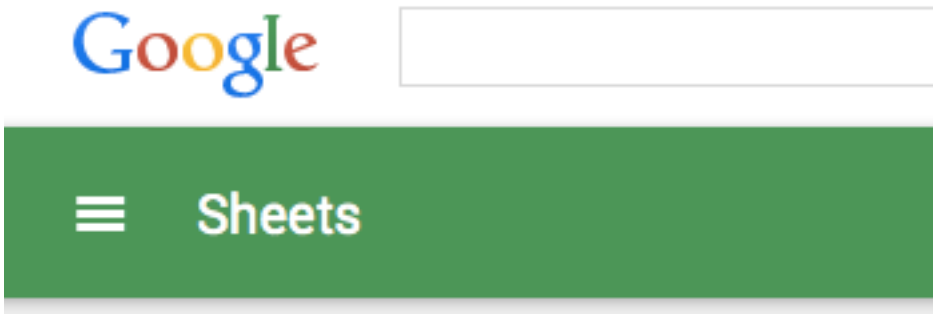
Insert; chart; scatterplot



Excel has automatic “fit line” function



Or, use Google sheets



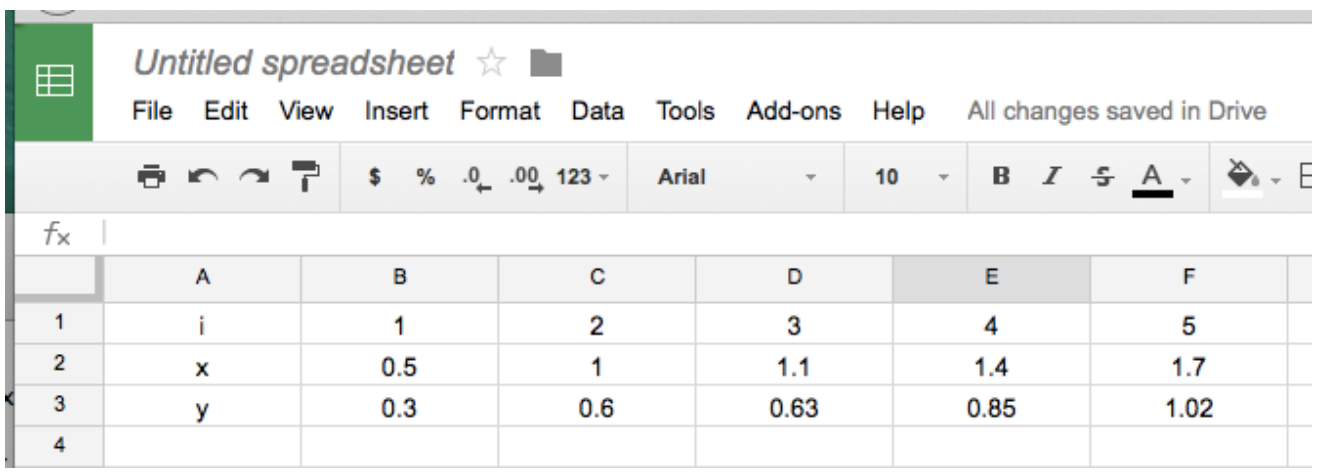
Here we show how to calculate best fits from scratch

- Copy the data from the Webwork question

Find the value of a which minimizes $f(s) = \sum_{i=1}^{17}(y_i - ax_i))^2$ where the data points (x_i, y_i) are given in the table below.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
x	0.5	1	1.1	1.4	1.7	1.8	2.3	2.4	2.6	3.2	3.4	3.5	3.6	3.9	4.4	4.6	4.9
y	0.3	0.6	0.63	0.85	1.02	1.05	1.35	1.46	1.48	1.99	2.06	2.05	2.13	2.22	2.58	2.84	2.85

Paste into Google sheets (do not retype!)



Pick some value of *a* and compute the residuals

	A	B	C	D	
1	i	1	2	3	
2	x	0.5	1	1.1	
3	y	0.3	0.6	0.63	0
4					
5	a=				
6	1				
7	residuals	-0.2	-0.4	-0.47	
8	residual^2	0.04	0.16	0.2209	
9	SSR=	25.7468			
10					

Residuals

	A	B
1	i	1
2	x	0.5
3	y	0.3
4		
5	a=	
6	1	
7	residuals	=B3-A6*B2

Square residuals

f_x =B7^2

	A	B
1	i	1
2	x	0.5
3	y	0.3
4		
5	a=	
6	1	
7	residuals	-0.2
8	residual^2	=B7^2

Sum of Square Residuals (SSR)

f_x =SUM(B8:R8)

	A	B	C
1	i	1	2
2	x	0.5	1
3	y	0.3	0.6
4			
5	a=		
6	1		
7	residuals	-0.2	-0.4
8	residual^2	0.04	0.16
9	SSR=	=SUM(B8:R8)	

Compute slope for best fit line y=*a*x

$ % .0 .00 123 Arial

fx =B13/B14

	A	B	C
1	i	1	2
2	x	0.5	1
3	y	0.3	0.6
4			
5	a=		
6	1		
7	residuals	-0.2	-0.4
8	residual^2	0.04	0.16
9	SSR=	25.7468	
10			
11	(xi*yi)	0.15	0.6
12	xi*2	0.25	1
13	sum(x1*yi)	91.968	
14	sum(xi)^2	155.11	
15	a=	=B13/B14	
16			

Compute slope for best fit line y=*a*x

	A	B
1	i	1
2	x	0.5
3	y	0.3

- $x_i\ y_i$
- x_i^2
- $\Sigma x_i\ y_i$
- $\Sigma\ x_i^2$
- $$a = \frac{\sum y_i x_i}{\sum x_i^2}$$
- (This is best slope)

11	(xi*yi)	=B2*B3
12	xi*2	=B2^2
13	sum(xi*yi)	=SUM(B11:R11)
14	sum(xi)^2	=SUM(B12:R12)

13	sum(xi*yi)	91.968
14	sum(xi)^2	155.11
15	a=	=B13/B14

The SSR should be small for the best fit line

- Large SSR for arbitrary value of *a*:

5	a=	
6	1	
7	residuals	-0.2
8	residual^2	0.04
9	SSR=	25.7468

- Much smaller SSR when we plug in the a value we found.

5	a=	
6	=B15	
7	residuals	0.003539423635
8	residual^2	0.000012527519 0
9	SSR=	0.0430274257

Plot the data

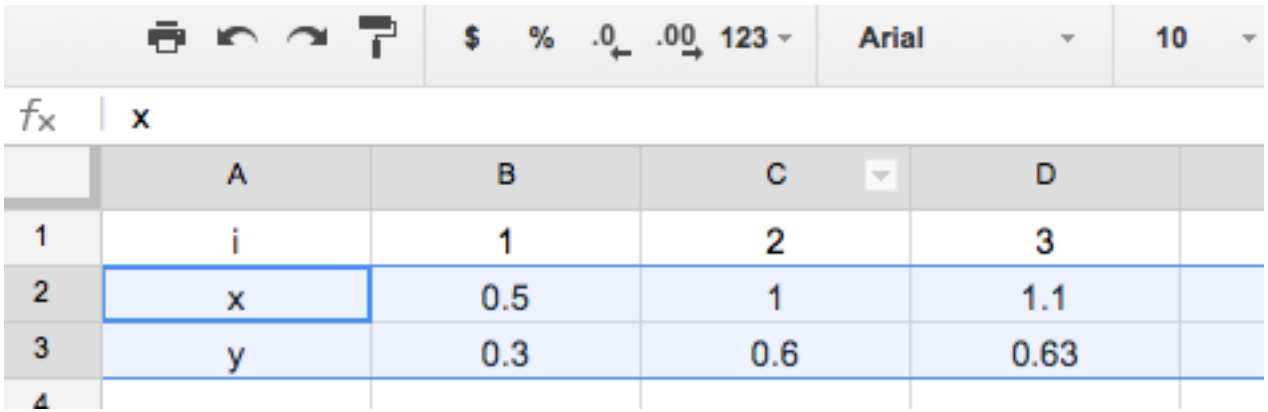
- Make a scatter plot of the data and get Google sheets to fit a best line to it.

$$y = ax + b$$

- Note: we can also do this ourselves by calculating the quantities a and b from the data

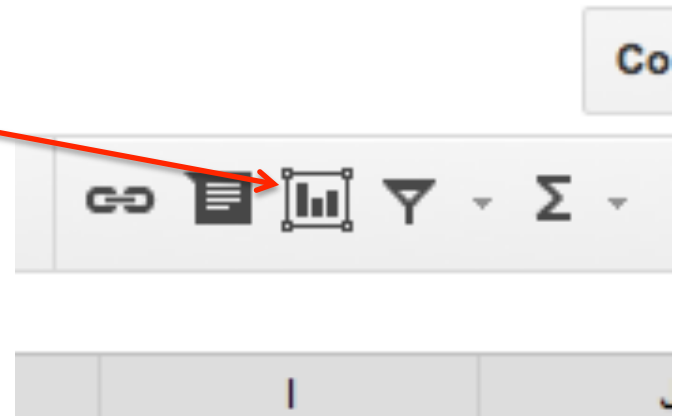
Make a scatter plot

- Highlight the cells with data, including labels



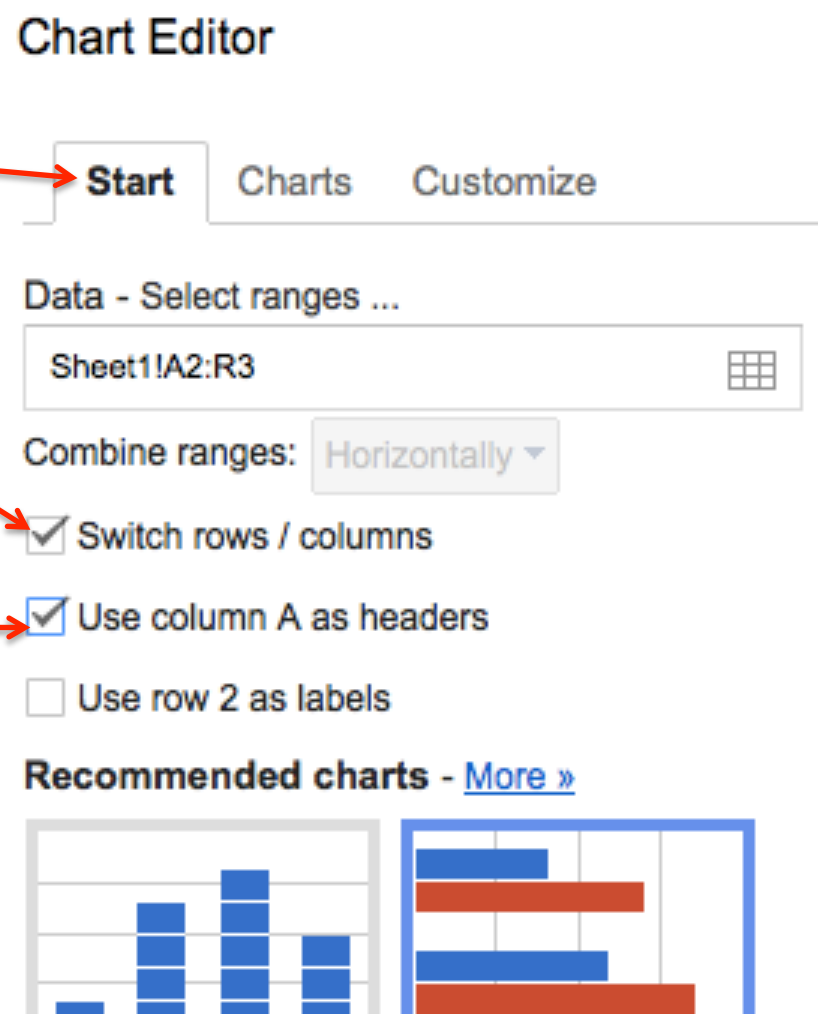
	A	B	C	D
1	i	1	2	3
2	x	0.5	1	1.1
3	y	0.3	0.6	0.63
4				

- Insert chart:

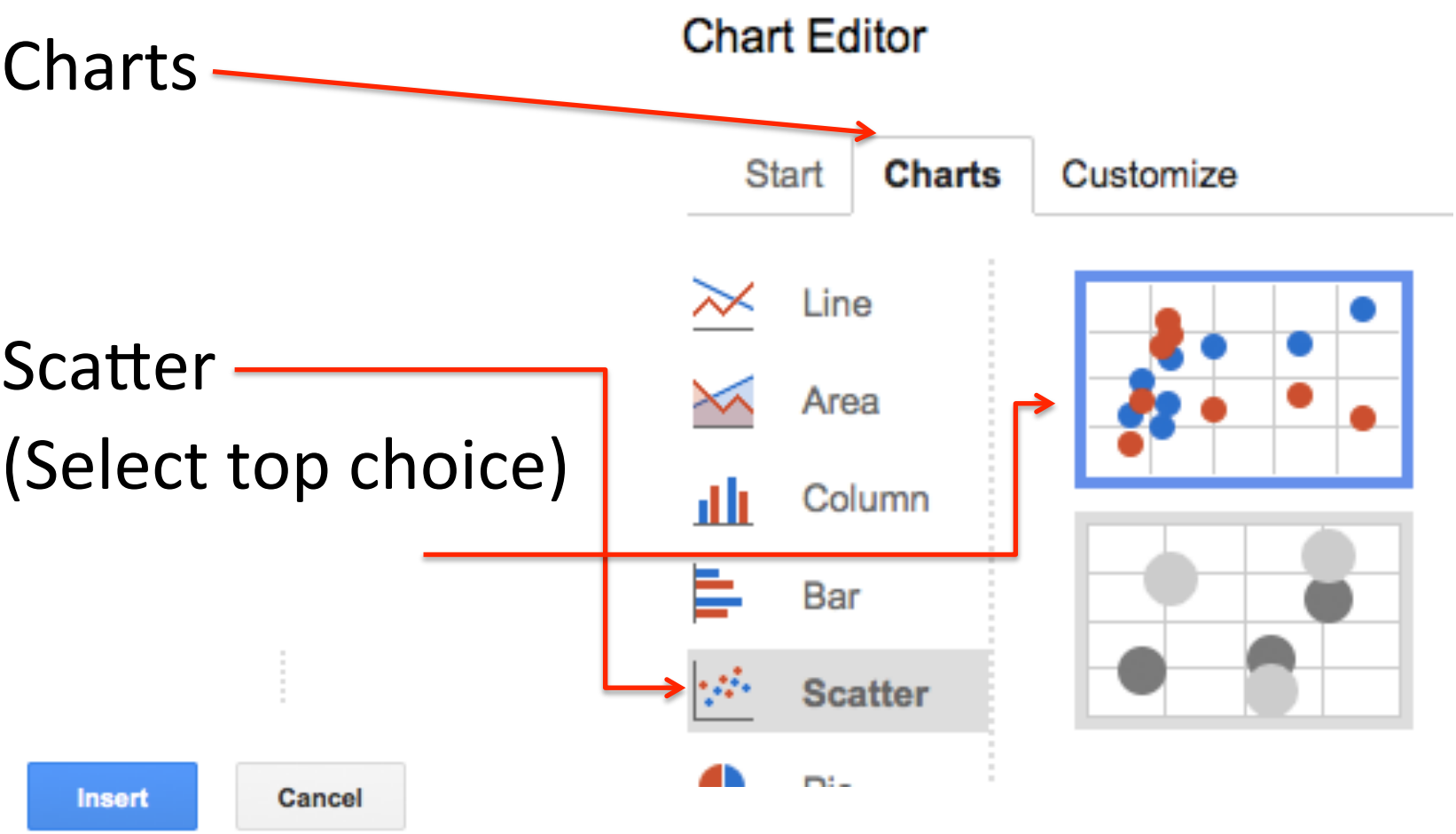


Format chart

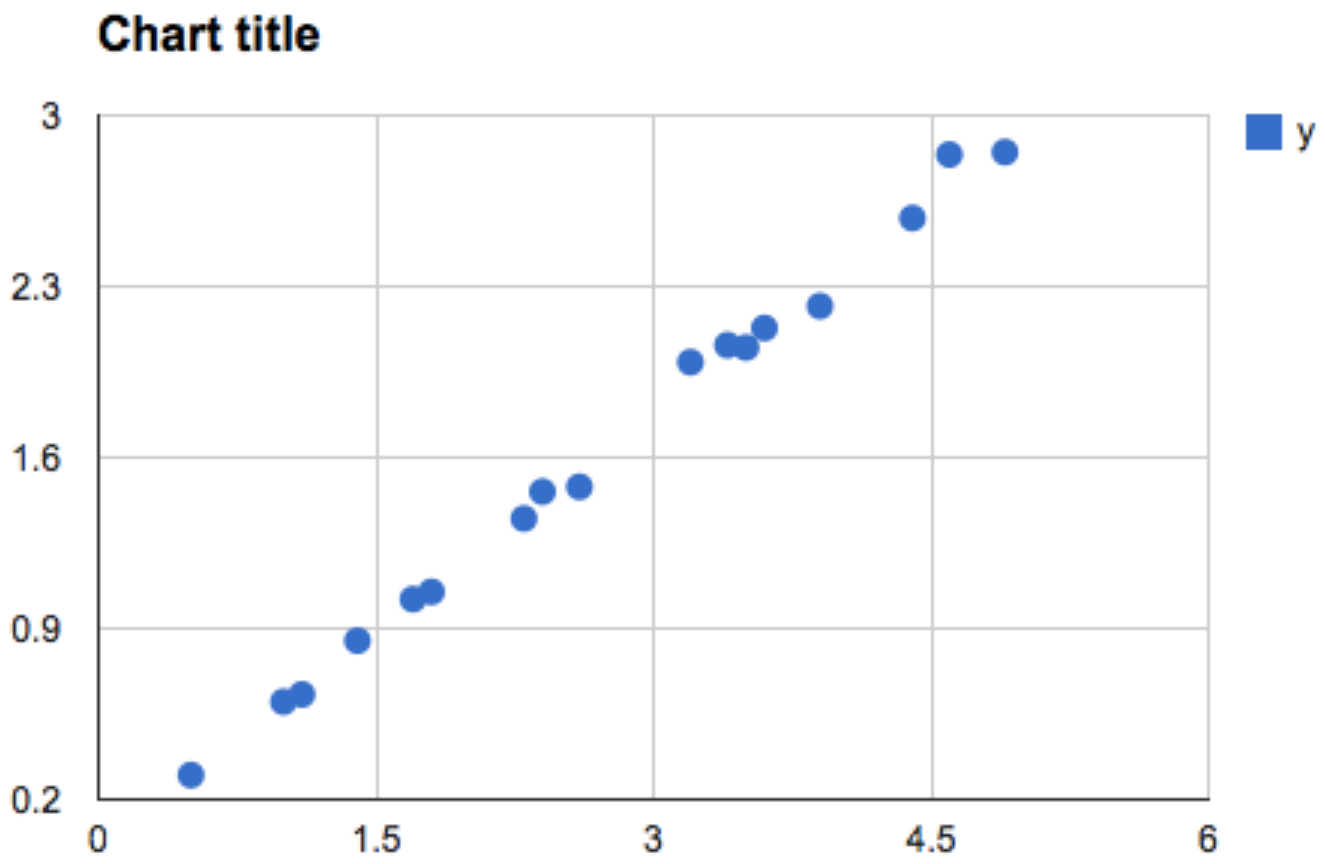
- Start
- Switch rows and columns
- Use column A as headers



- Charts
- Scatter
- (Select top choice)
-

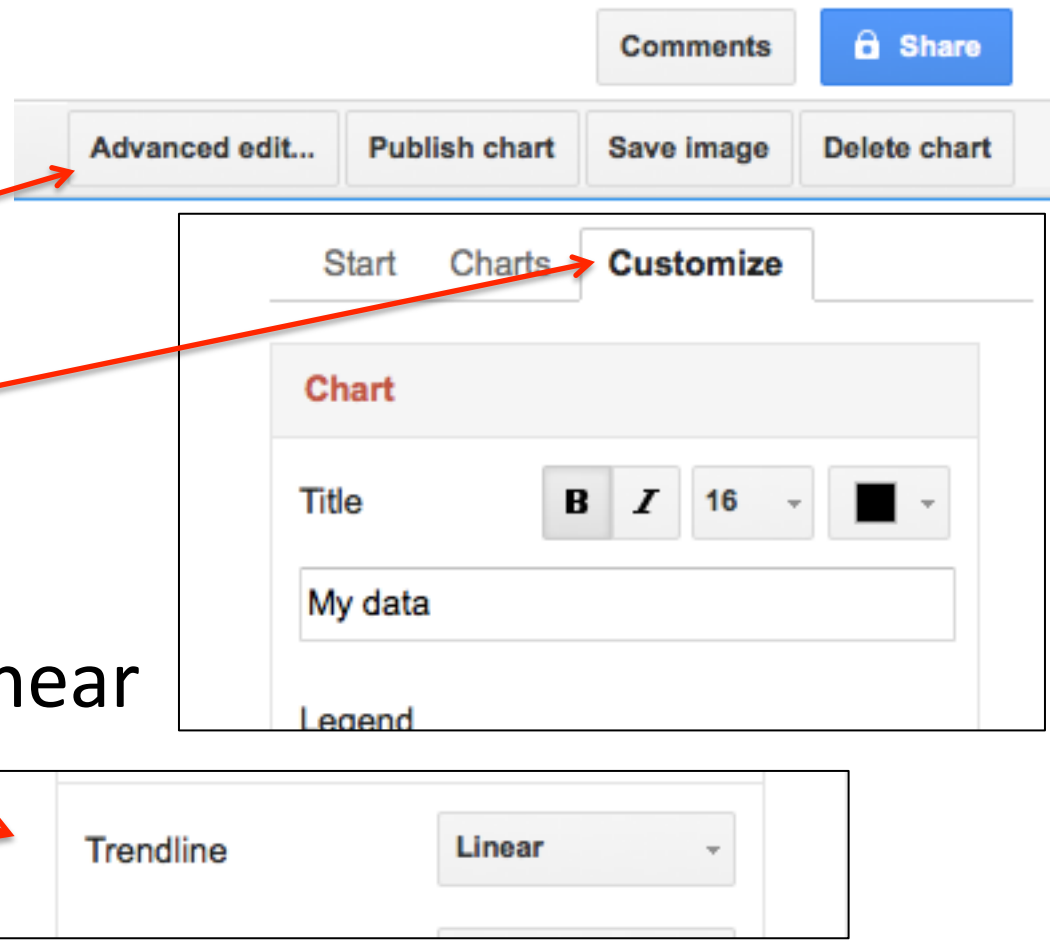


Data plot will appear

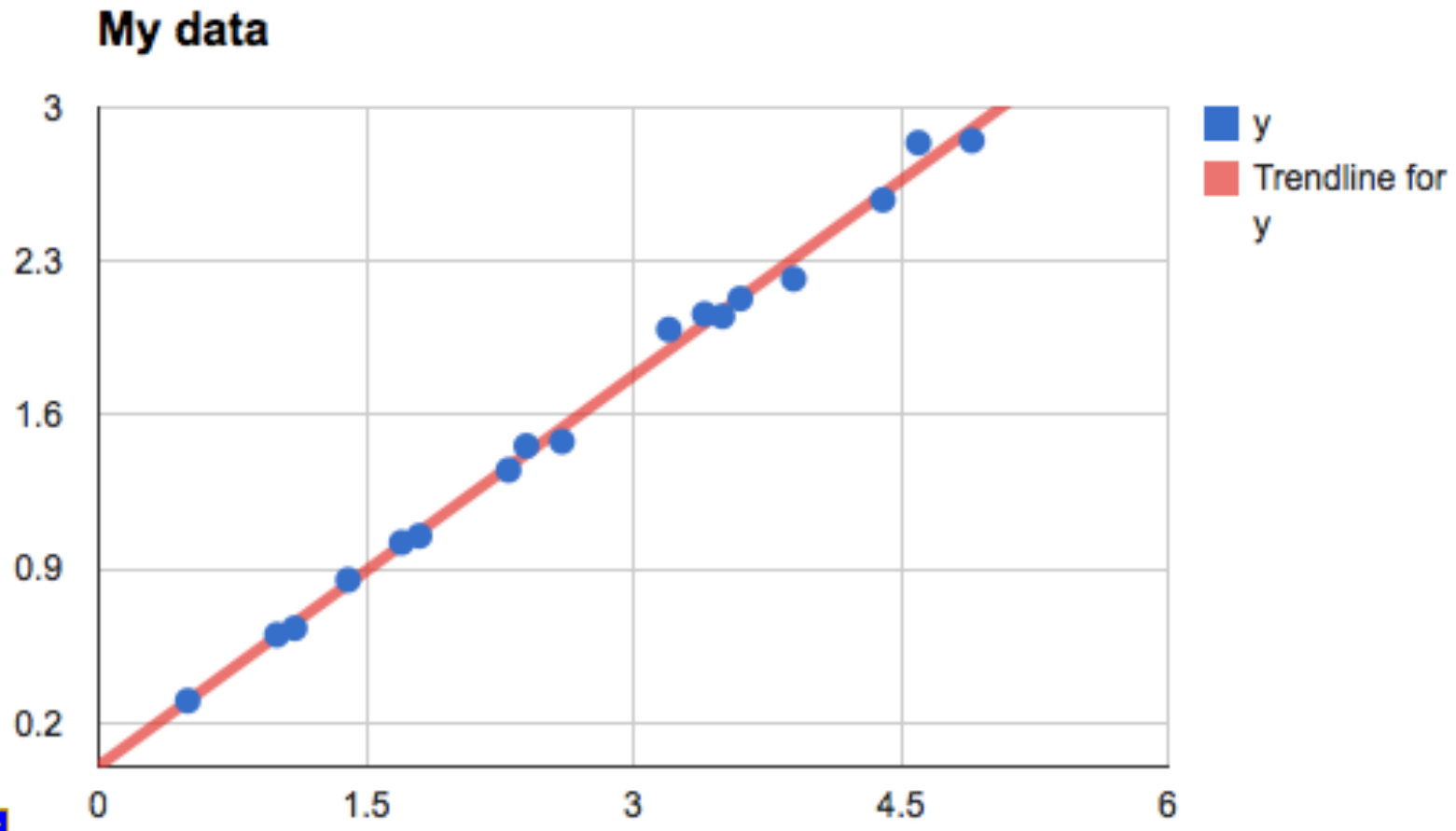


Add trendline

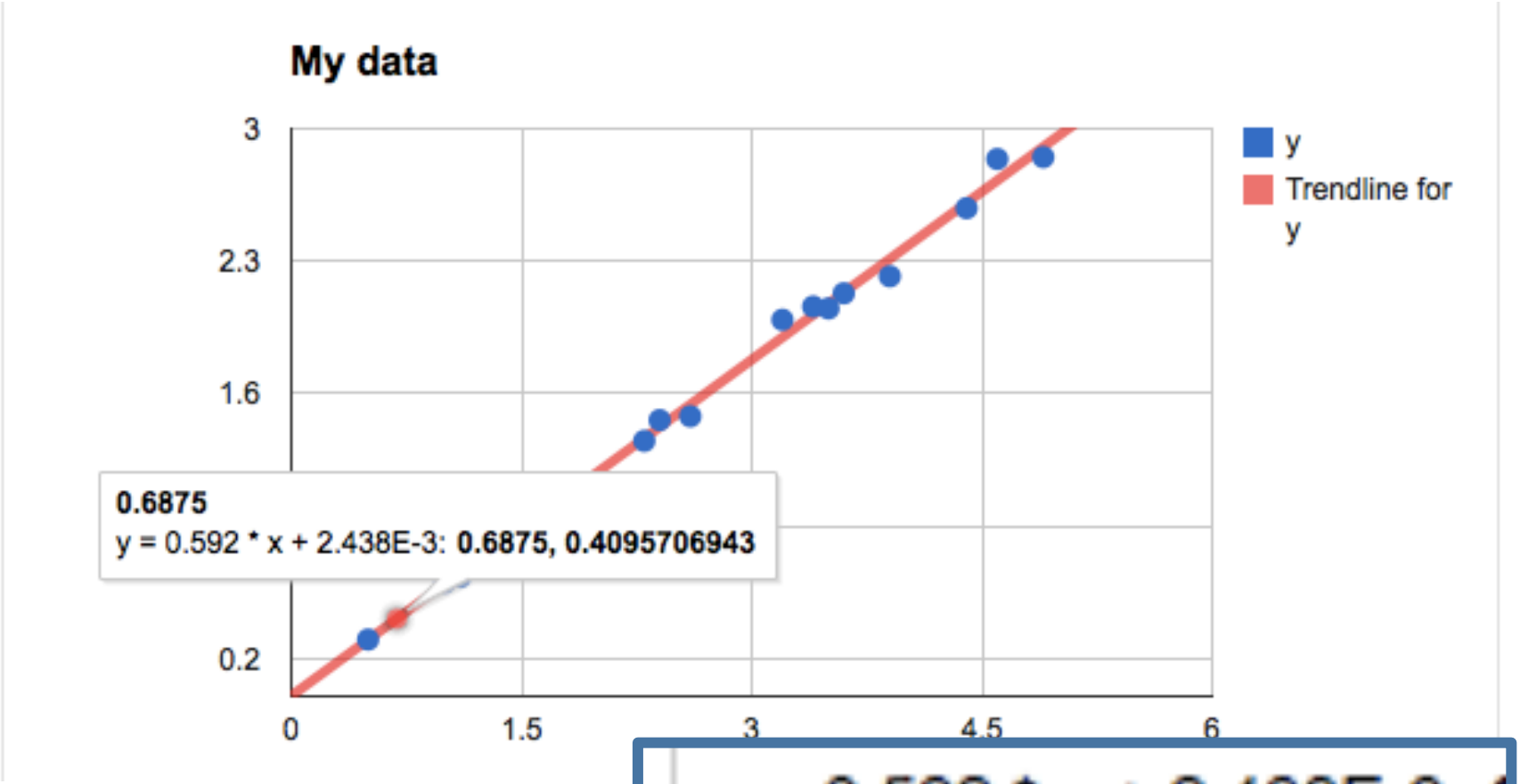
- {Control click} on chart
- Advanced edit
- Customize
- Scroll down menu
- Select Trendline; linear

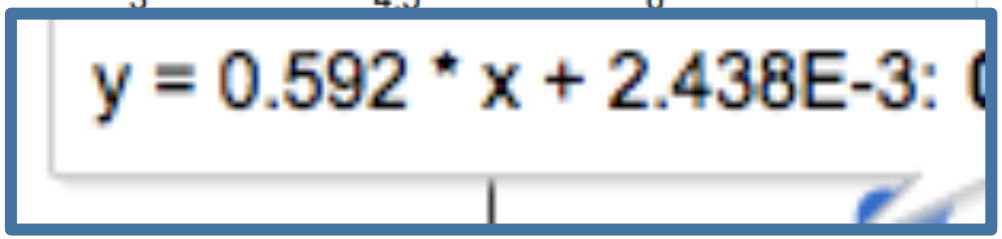


Trendline will appear



Clicking on line displays trendline





Line agrees with our own calculations

- Calculated values

18	Pav=	5.409882353
19	X^2av=	9.124117647
20	xbar=	2.723529412
21	ybar=	1.615294118
22	a=	0.5921935196
23	b=	0.00243764954
24		

$$P_{avg} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

$$X^2_{avg} = \frac{1}{n}\sum_{i=1}^{n} (x_i^2)$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$a = \frac{P_{avg} - \bar{x}\bar{y}}{X^2_{avg} - \bar{x}^2}$$

$$b = \bar{y} - a\bar{x}$$

Calculations

- Cells:

18	Pav=	=B13/17
19	X^2av=	=B14/17
20	xbar=	=SUM(B2:R2)/17
21	ybar=	=SUM(B3:R3)/17
22	a=	=(B18-B20*B21)/(B19-B20^2)
23	b=	=B21-B22*B20
24		

$$a = \frac{P_{avg} - \bar{x}\bar{y}}{X^2_{avg} - \bar{x}^2}$$

$$b = \bar{y} - a\bar{x}$$

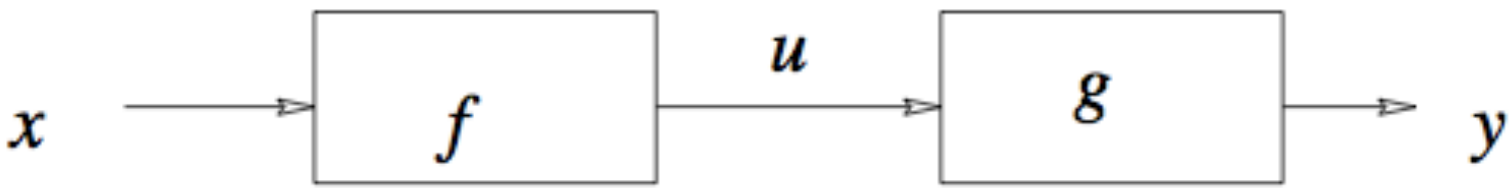
Chain Rule - Intro

Section 8.1 Learning goals

1. Understand the concept of function composition and be able to express a composite function in terms of the underlying composed functions.
2. Understand the chain rule of differentiation and be able to use it to find the derivative of a composite function.

Function composition

- A function y=g(f(x))



What is the correct decomposition?

- If the function is $f(x) = (3x^2 - 1)^7$

Then the decomposition is

A. $a(x) = 3x^2 - 1, b(x) = (3x^2 - 1)^7, f(x) = b(a(x))$

B. $a(x) = 3x^2 - 1, b(x) = x^7, f(x) = a(b(x))$

C. $a(x) = 3x^2 - 1, b(x) = (3x^2 - 1)^7, f(x) = a(b(x))$

D. $a(x) = 3x^2 - 1, b(x) = x^7, f(x) = b(a(x))$

E. $a(x) = 3x^7, b(x) = x^2 - 1, f(x) = a(b(x))$

Here are values of f(x) at a few points

x	-3	-2	-1	0	1	2	3
$f(x)$	0	-3	1	2	3	1	-2

Then f(f(3)) is equal to:

A. 3

B. -3

C. 0

D. -2

E. 2

(1) Function composition

If $f(x) = x^3 + x^2$ and $g(x) = \sqrt{x}$ then $f(g(x))$ is

(A) $\sqrt{x^3 + x^2}$, (B) $\sqrt{x^3} + x^2$, (C) $x^{3/2} + x$, (D) $x^2\sqrt{x+1}$,

(E) None of the above

(2) Function composition

If $f(x) = x^3 + x^2$ and $g(x) = \sqrt{x}$ then $g(f(x))$ is

(A) $\sqrt{x^3 + x^2}$, (B) $\sqrt{x^3} + x^2$, (C) $x^{3/2} + x$, (D) $x^2\sqrt{x+1}$,

(E) None of the above

Chain Rule of differentiation

- If y = g(u) and u = f(x) are both differentiable functions and y = g(f(x)) is the composite function, then the **chain rule** of differentiation states that

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Coffee Budget

- Rate of increase of money spent on coffee

=

Rate of increase of price per cup

times

Rate of increase of cups I drink

(3) Chain Rule example

Chain Rule: The derivative of the function $y = f(x) = \sqrt{x^2 + a^2}$ is

$$\text{(A) } f'(x) = \frac{1}{\sqrt{x^2+a^2}}, \quad \text{(B) } f'(x) = \frac{1}{2\sqrt{x^2+a^2}}, \quad \text{(C) } f'(x) = \frac{x}{2\sqrt{x^2+a^2}},$$

$$\text{(D) } f'(x) = \frac{2x}{\sqrt{x^2+a^2}}, \quad \text{(E) } f'(x) = \frac{x}{\sqrt{x^2+a^2}},$$

Solution

- The function is $f(x) = \sqrt{x^2 + a^2}$. Let $u = x^2 + a^2$.

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2} \cdot 2x = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2 + a^2}}$$

(4) Chain rule example

Compute the derivative of

$$y = f(x) = \frac{x}{\sqrt{x^2 + d^2}},$$

)

(4) Chain rule example

The derivative of $y = f(x) = \dfrac{x}{\sqrt{x^2 + d^2}}$, is

(A) $d^2/(x^2 + d^2)^{3/2}$

(B) $x^2/(x^2 + d^2)^{1/2}$

(C) $d^2/(x^2 + d^2)^{1/2}$

(D) $x^2/(x^2 + d^2)^{3/2}$

(E) None of the above

Chain rule example (2)

The derivative of $y = f(x) = \dfrac{x}{\sqrt{x^2 + d^2}}$, is:

Quotient rule:

$$\frac{dy}{dx} = \frac{[x]' \cdot \sqrt{x^2 + d^2} - [\sqrt{x^2 + d^2}]' \cdot x}{(\sqrt{x^2 + d^2})^2}$$

$$\frac{dy}{dx} = \frac{1 \cdot \sqrt{x^2 + d^2} - [\frac{1}{2} \cdot 2x \cdot (x^2 + d^2)^{-1/2}] \cdot x}{(x^2 + d^2)}$$

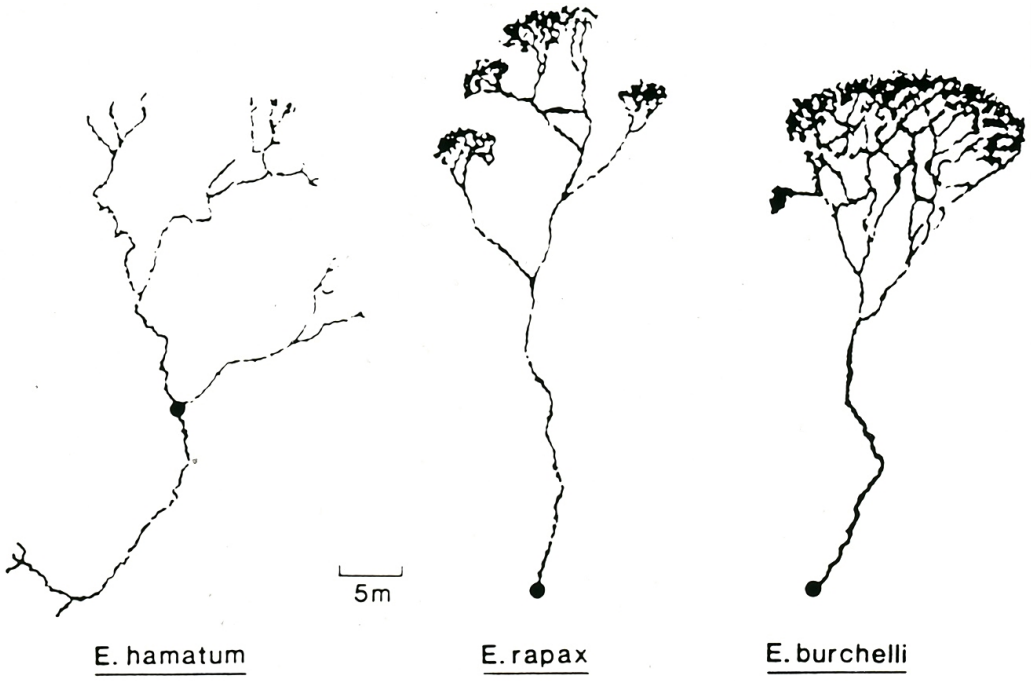
$$= \frac{x^2 + d^2 - x^2}{(x^2 + d^2)^{1/2}(x^2 + d^2)} = \frac{d^2}{(x^2 + d^2)^{3/2}}$$

Chain rule applied to optimization

Section 8.2 Learning goals

1. Read and follow the derivation of each optimization model.
2. Be able to carry out the calculations of derivatives appearing in the problems (using the chain rule)
3. Using optimization, find each critical point and verify its type.
4. Understand and be able to explain the interpretation of the mathematical results.

Ant trails



J.L. Deneubourg et al.

Army Ant Raid Patterns

I love ants! (but not in my kitchen)

Modelling the Formation of Trail Networks by Foraging Ants

JAMES WATMOUGH† AND LEAH EDELSTEIN-KESHET‡

J. theor. Biol. (1995) **176,** 357–371

FIG. 3 (a and b).

UBC Math 102

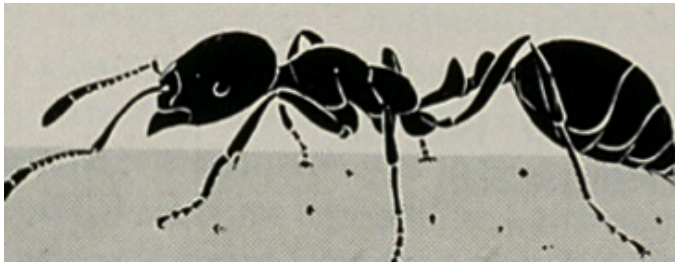
Ant trails

Ants: trail forming in an arena

- https://www.youtube.com/watch?v=-lz97WTM7aU

Ants: pheromone markings

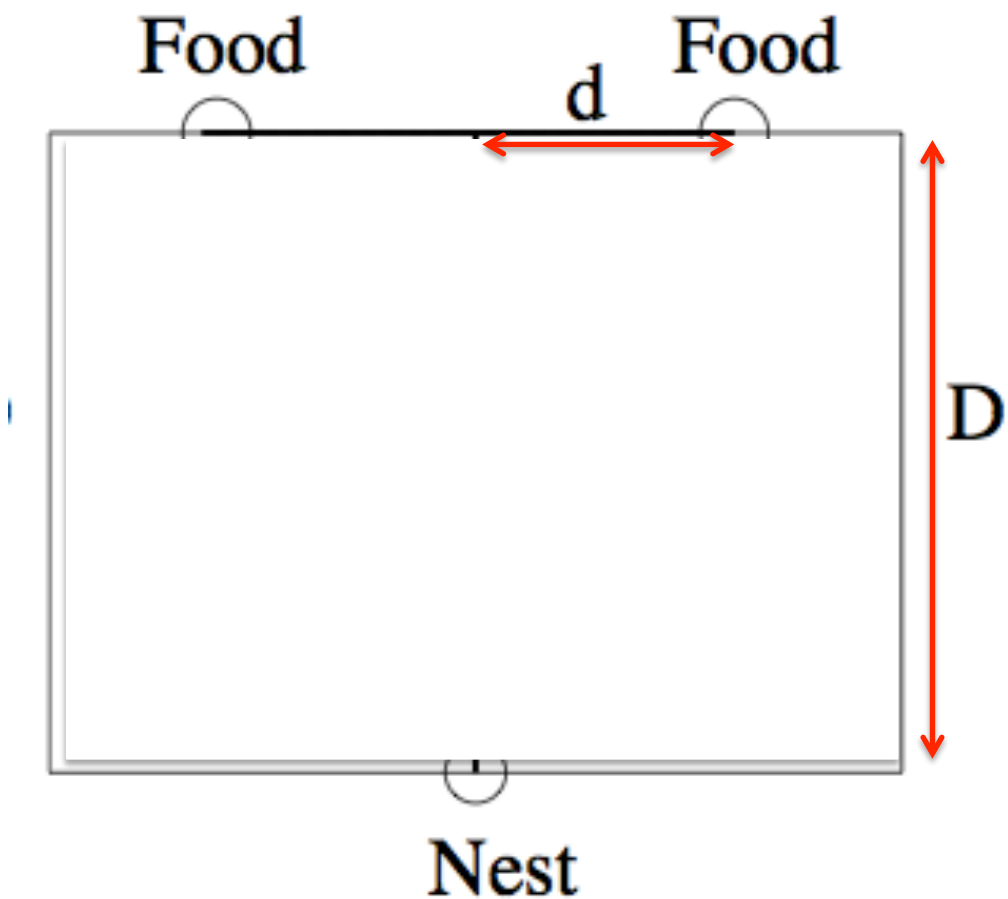
- https://www.youtube.com/watch?v=tAe3PQdSqzg



Ant trails

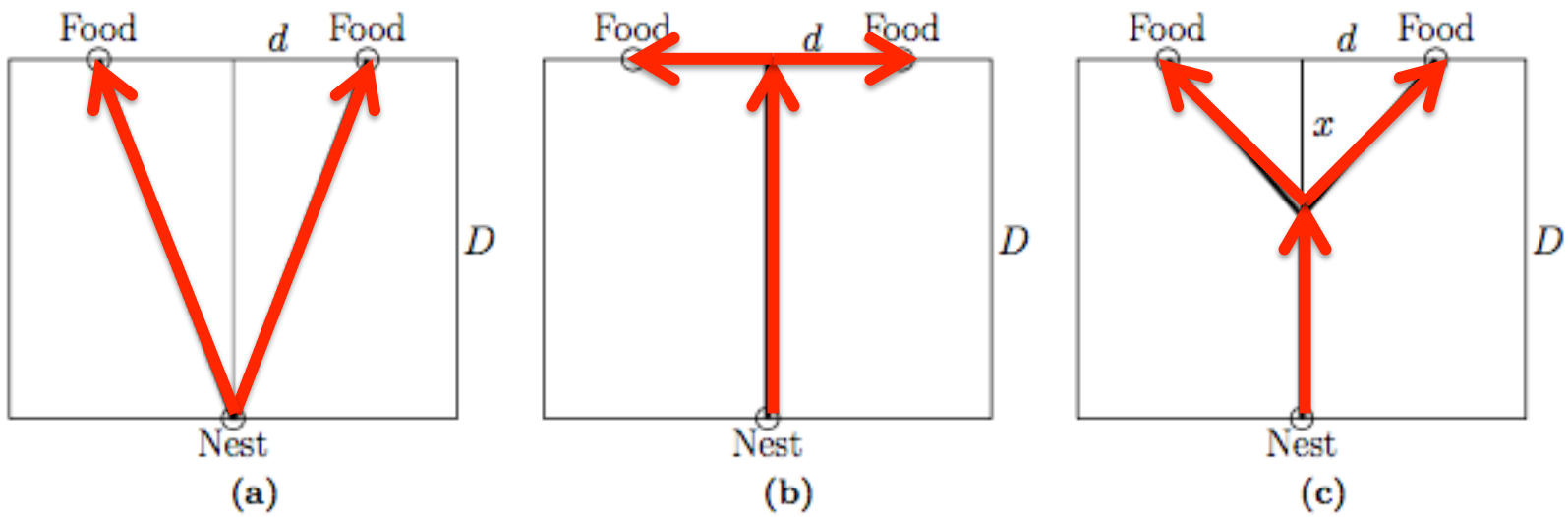
- Ants can find the shortest route that connects their nest to a food source.
- each ant secretes a chemical **pheromone** that other ants will follow.

Find the minimum trail length connecting nest to two food sources



Some possible trails

- We can think of several ways to get from the nest to the food



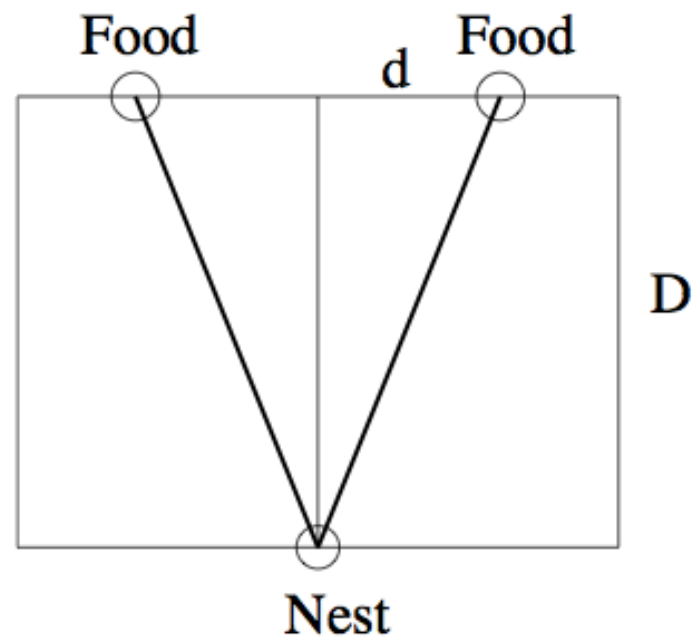
(1) A V-shaped ant trail

- What is the total length of this V shaped trail?

(A) $D+2d$, (B) $2D+d$,

(C) $2\sqrt{D+d}$, (D) $2\sqrt{D^2+d^2}$,

(E) $2(D+d)$,



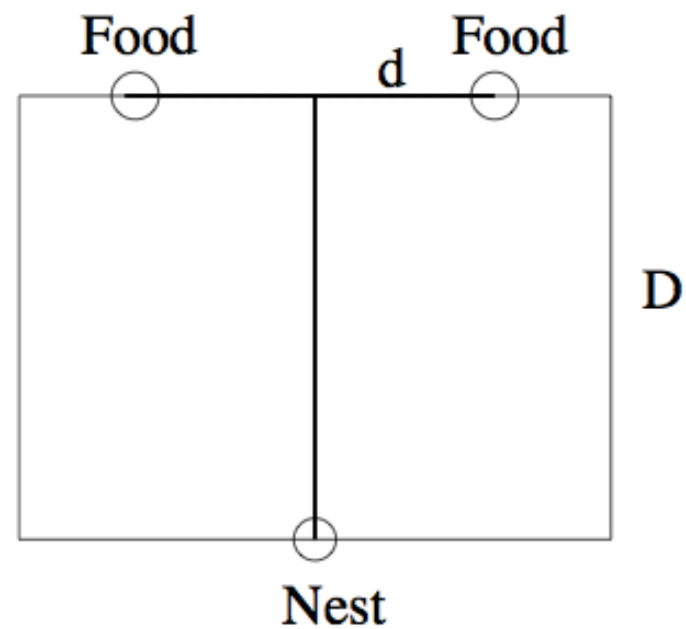
(2) A T-shaped ant trail:

- What is the total length of this T shaped trail?

(A) $D+2d$, (B) $2D+d$,

(C) $2\sqrt{D+d}$, (D) $2\sqrt{D^2+d^2}$,

(E) $2(D+d)$,



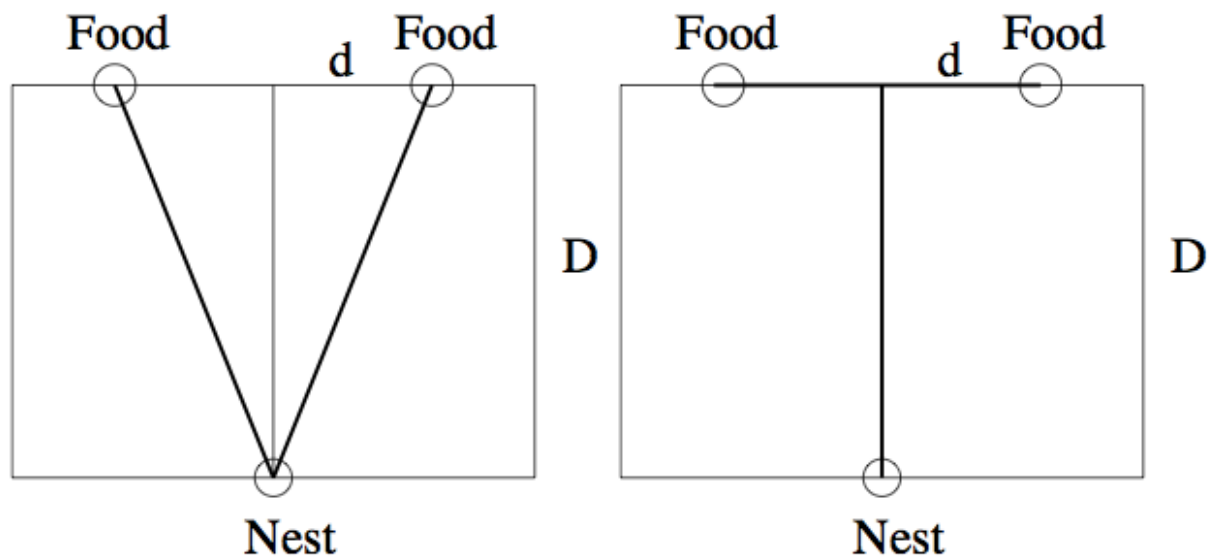
(3) Which path is shorter?

(A) The V path.

(B) The T path.

(C) They are the same length.

(D) It depends on d and D.



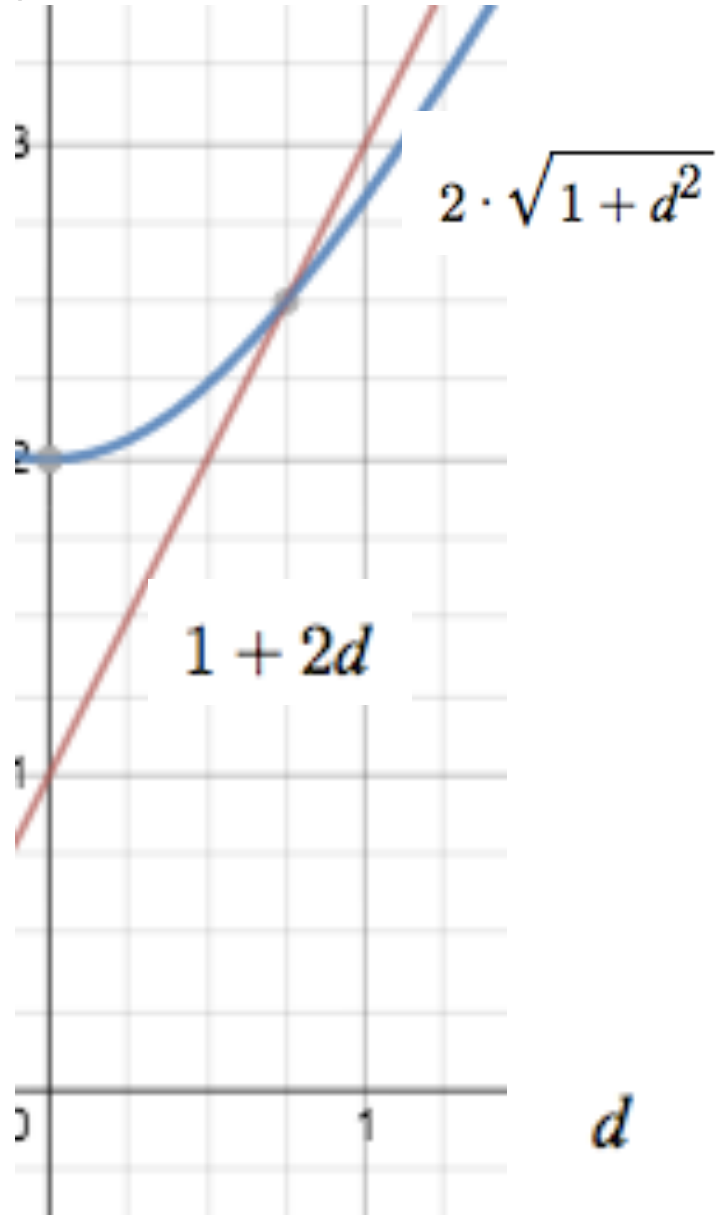
It depends on d and D!

Example: suppose D=1

The length of

T path V path →

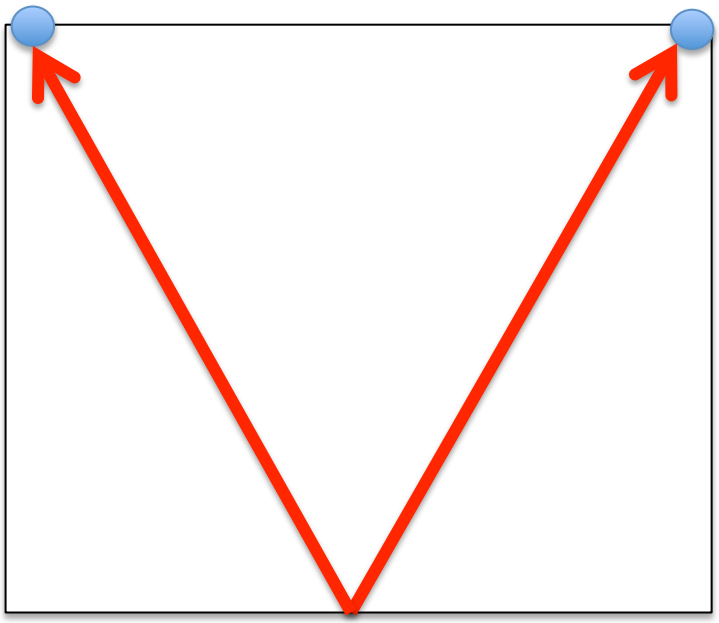
Largest depends on d!

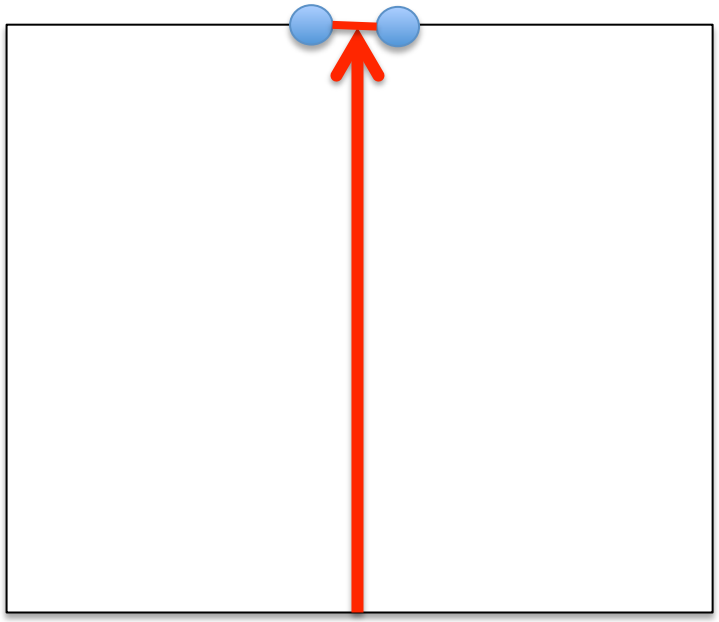


Small vs large d

- large d

small d





(4) Y shaped trail

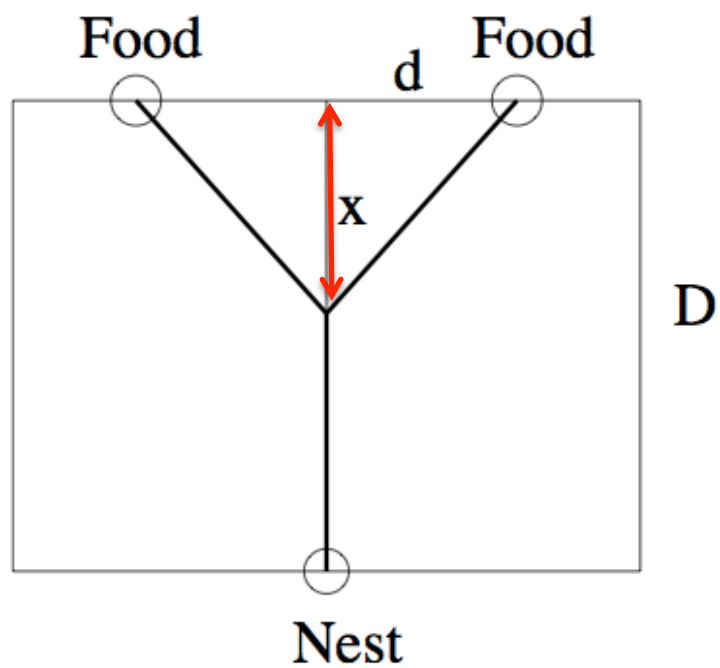
What is the total length of the trail shown?

(A) $D + 2\sqrt{d^2 + x^2}$, (B) $D - x + 2\sqrt{d^2 + x^2}$,

(C) $D - x + \sqrt{d^2 + x^2}$

(D) $D(1 - d/x)$,

(E) $D(1 - x/d)$,



(5) The Ant trail problem

We have now found the length L_Y of the Y-shaped path. What are we going to do next?

(A) Minimize L_Y with respect to d.

(B) Minimize L_Y with respect to x.

(C) Minimize L_Y with respect to D.

(D) Maximize L_Y with respect to x.

(E) Maximize L_Y with respect to d.

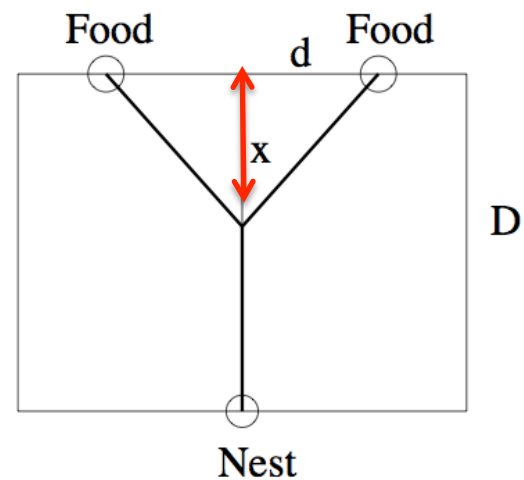
(6) What is the range of values of x ?

$$L_y(x) = D - x + 2\sqrt{d^2 + x^2}$$

(A) $0 \leq x$, (B) $0 \leq x \leq d$,

(C) $0 \leq x \leq D$, (D) $0 \leq x \leq 2d$,

(E) $-D \leq x \leq D$,



Derivative

Use the chain rule to find the derivative of this function:

$$L(x) = D - x + 2\sqrt{d^2 + x^2}$$

$$L(x) = D - x + 2\sqrt{d^2 + x^2}$$

(7) Derivative

The derivative of the function for the Y-shaped path length is

(A) $L'(x) = -1 + \dfrac{1}{2\sqrt{x^2 + d^2}}$,

(B) $L'(x) = -1 + \dfrac{1}{\sqrt{x^2 + d^2}}$,

(C) $L'(x) = -1 + \dfrac{x}{\sqrt{x^2 + d^2}}$

(D) $L'(x) = -x + \dfrac{x}{\sqrt{x^2 + d^2}}$,

(E) $L'(x) = -1 + \dfrac{2x}{\sqrt{x^2 + d^2}}$,

$$L'(x) = -1 + \frac{2x}{\sqrt{x^2 + d^2}}$$

(8) Critical Points

The critical points are:

$$\text{(A) } x = \frac{d}{3}, \quad \text{(B) } x = \frac{d}{\sqrt{3}},$$

$$\text{(C) } x = \frac{d}{2}, \quad \text{(D) } x = \frac{d}{\sqrt{2}},$$

$$\text{(E) } x = \frac{d}{\sqrt{5}},$$

Critical points

- Set $L'(x) = 0,$

$$-1 + 2\frac{x}{\sqrt{x^2 + d^2}} = 0.$$

$$\sqrt{x^2 + d^2} = 2x$$

$$x^2 + d^2 = 4x^2$$

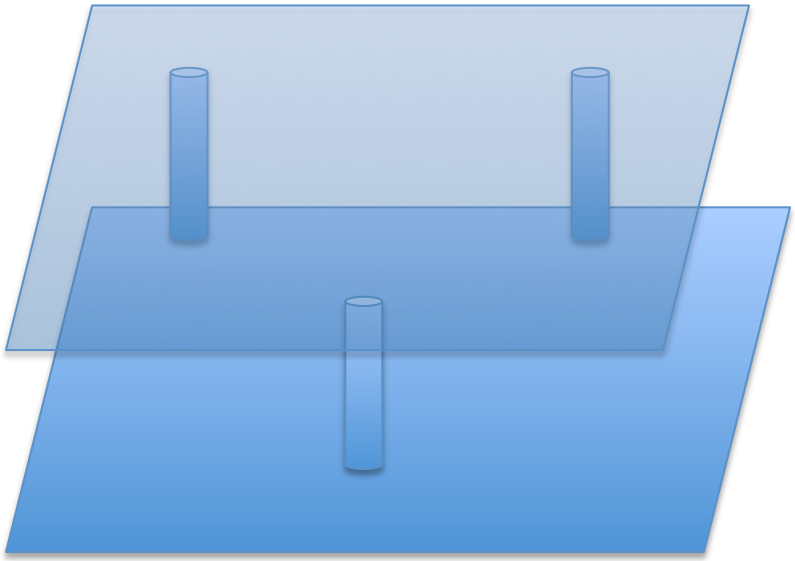
$$3x^2 = d^2 \quad \Rightarrow \quad x = \frac{d}{\sqrt{3}}.$$

Annother way to get the solution

Minimal surfaces (a.k.a. “soap bubbles”)

A solution by soap!

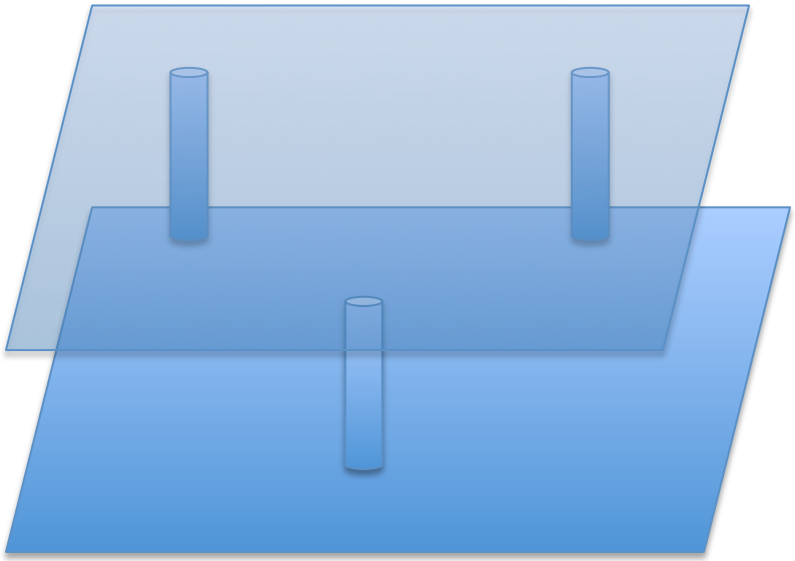
Two flat plexiglass plates with “pegs” representing the nest and food sources:



Idea: credit to Shawn Desaulniers

A solution by soap!

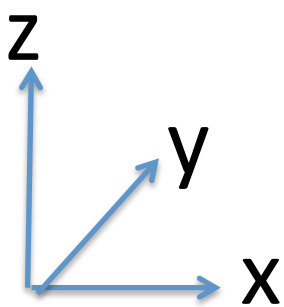
Soap bubbles form minimal surfaces (i.e. shapes that have the smallest possible surface area for given constraints:

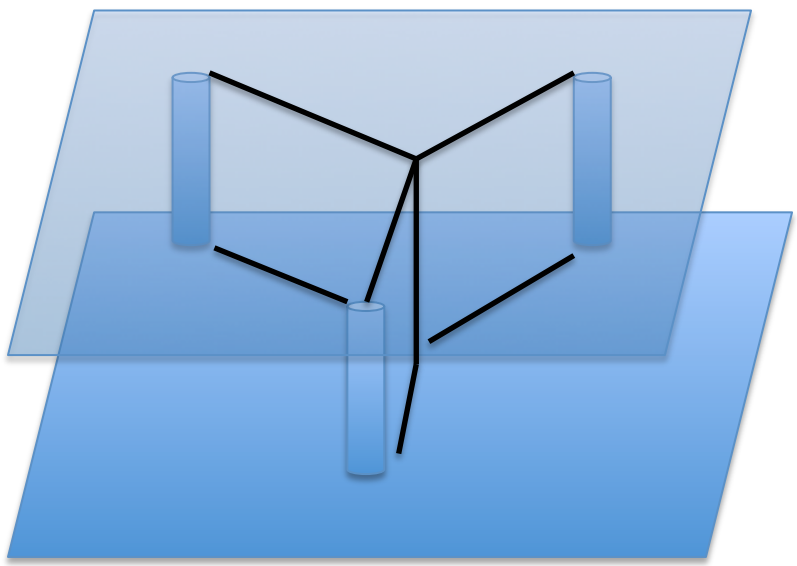


Here three pegs and 2 flat plates form the constraints that the soap surface has to meet.

A solution by soap!

The pegs and flat plates set a constant height in the z direction:





Hence the soap bubble selects a configuration with shortest length in the xy plane

Now time to finish the calculus problem (for practice)

(9) Are we done yet?

(A) No, we still need to find the length L(x) at the critical point.

(B) No we still need to find how far apart the food sources are, i.e. solve for d

(C) No, we need to check the type of critical point

(D) Yes, we are done

Checking via the second derivative

Find the second derivative of the function

$$L(x) = D - x + 2\sqrt{d^2 + x^2}$$

Recall that the first derivative is

$$L'(x) = -1 + \frac{2x}{\sqrt{x^2 + d^2}}$$

Solution:

See an earlier chain-rule derivative problem, and Example 8.5 in the M102 Notes.

$$L''(x) = 2\,\frac{d^2}{(x^2+d^2)^{3/2}}.$$

(So what does this tell us?)

(10) What does this tell us?

$$L''(x) = 2 \cdot \frac{d^2}{(x^2 + d^2)^{3/2}}.$$

(A) The second derivative can be either positive or negative, so the critical point could be of either type

(B) The second derivative is always positive so the critical point is a local minimum.

(C) The second derivative is always positive so the critical point is a local maximum.

Sketching the original function

Sketch the function

$$L(x) = (D - x) + 2\sqrt{d^2 + x^2}.$$

(Remember, this is an alternative to first or second derivative tests for diagnosing a critical point.)

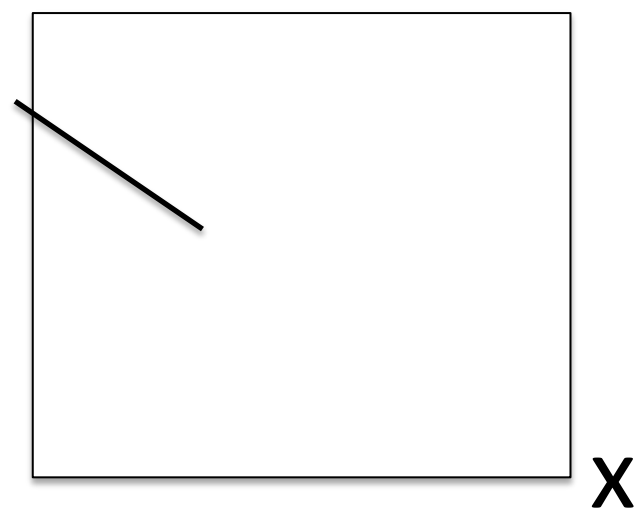
Sketching the original function

Sketch the function

$$L(x) = (D - x) + 2\sqrt{d^2 + x^2}.$$

- Small x: L≈(D-x)+2d = C-x

(straight line, slope -1)



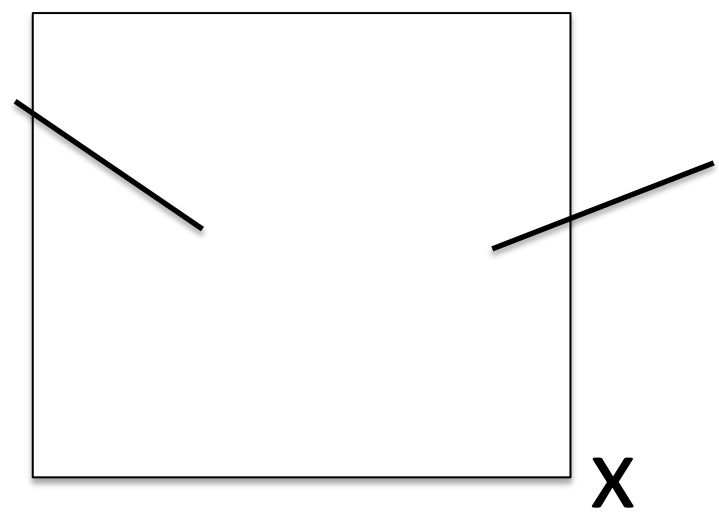
Sketching the original function

Sketch the function

$$L(x) = (D - x) + 2\sqrt{d^2 + x^2}.$$

- Small x: L≈(D-x)+2d = C-x
- Large x: L ≈(D-x)+2x =D+x

(straight line, slope +1)



Sketching the original function

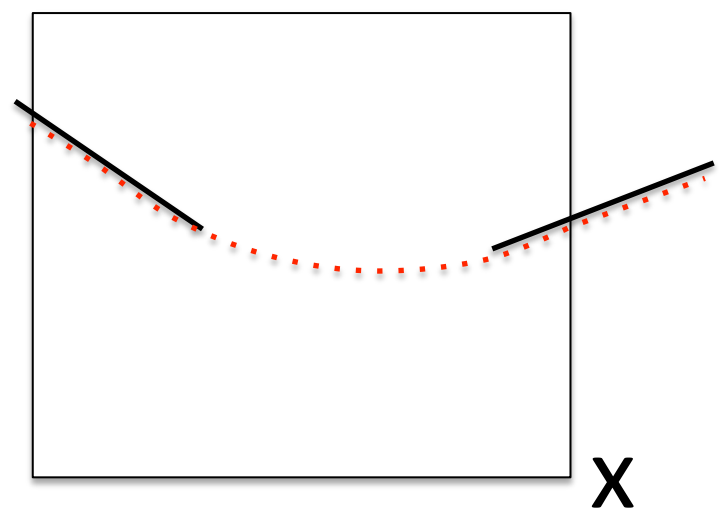
Sketch the function

$$L(x) = (D - x) + 2\sqrt{d^2 + x^2}.$$

- Small x: L≈(D-x)+2d = C-x
- Large x: L ≈(D-x)+2x =D+x

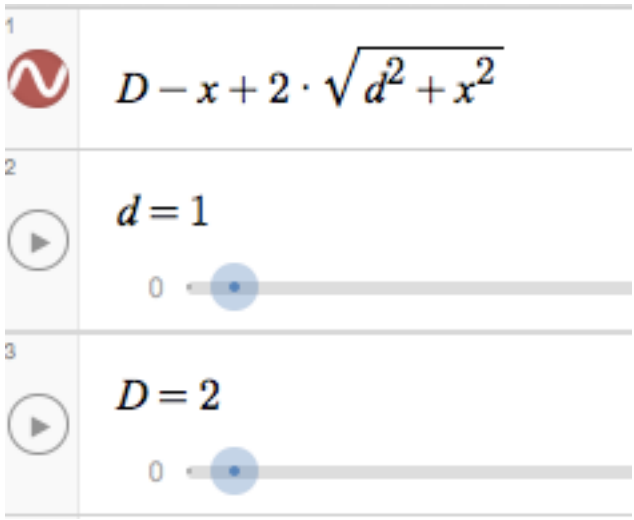
Connect these smoothly!

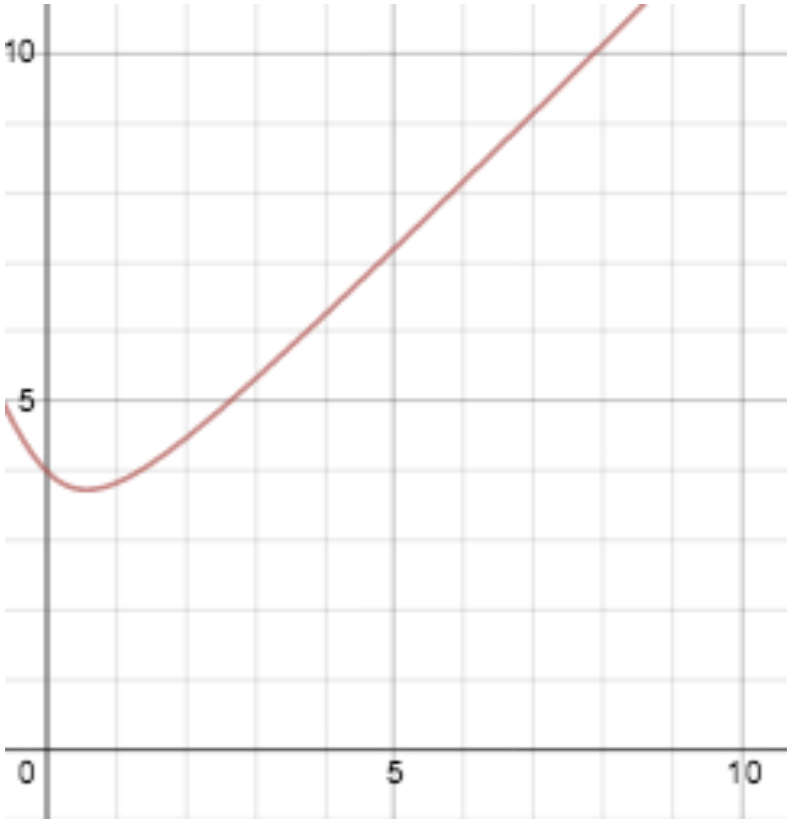
verifies a LOCAL MIN



Desmos

- Graph of the function with D=1



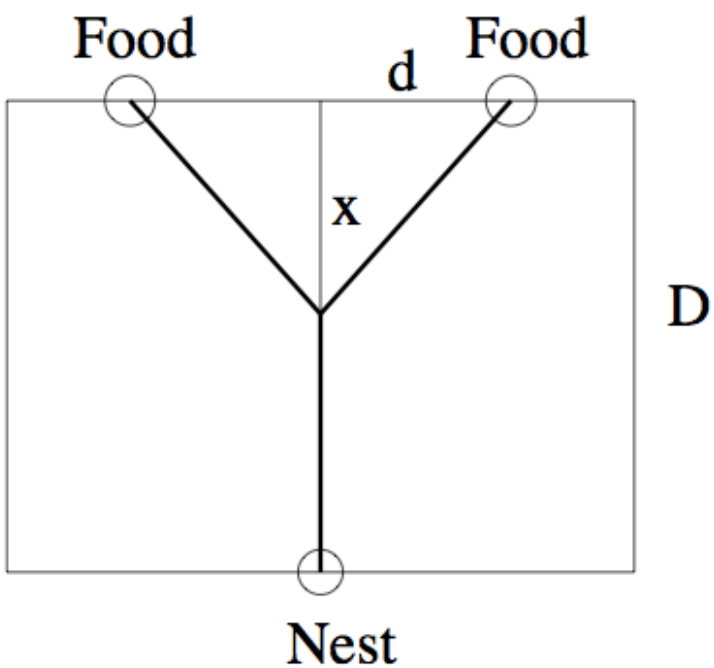


For further study: Angles

What angles does the Y trail form?

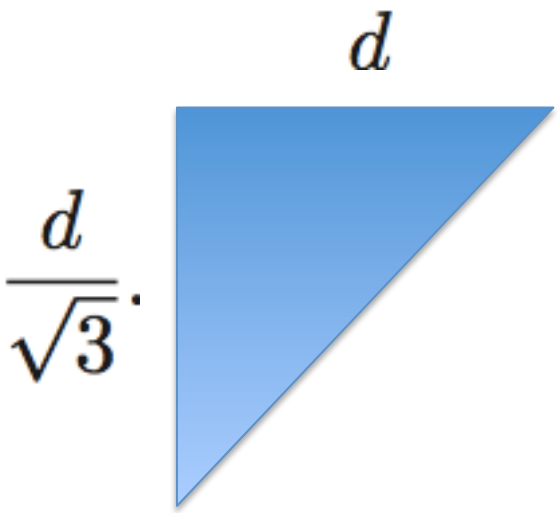
We found that

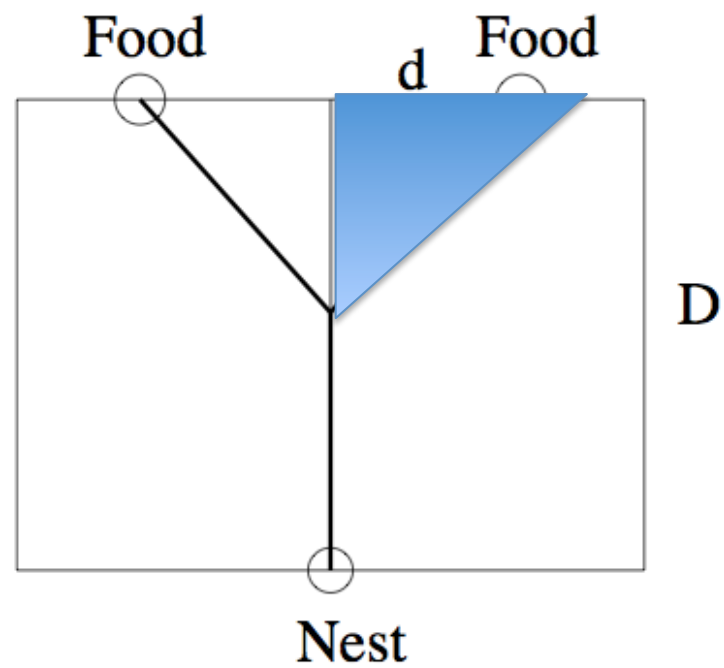
$$x = \frac{d}{\sqrt{3}}.$$



Angles

What angle does the Y trail form?





See if you can answer this for next time.

Answers

Data fitting

- 1 C
- 2 C
- 3 E
- 4 C

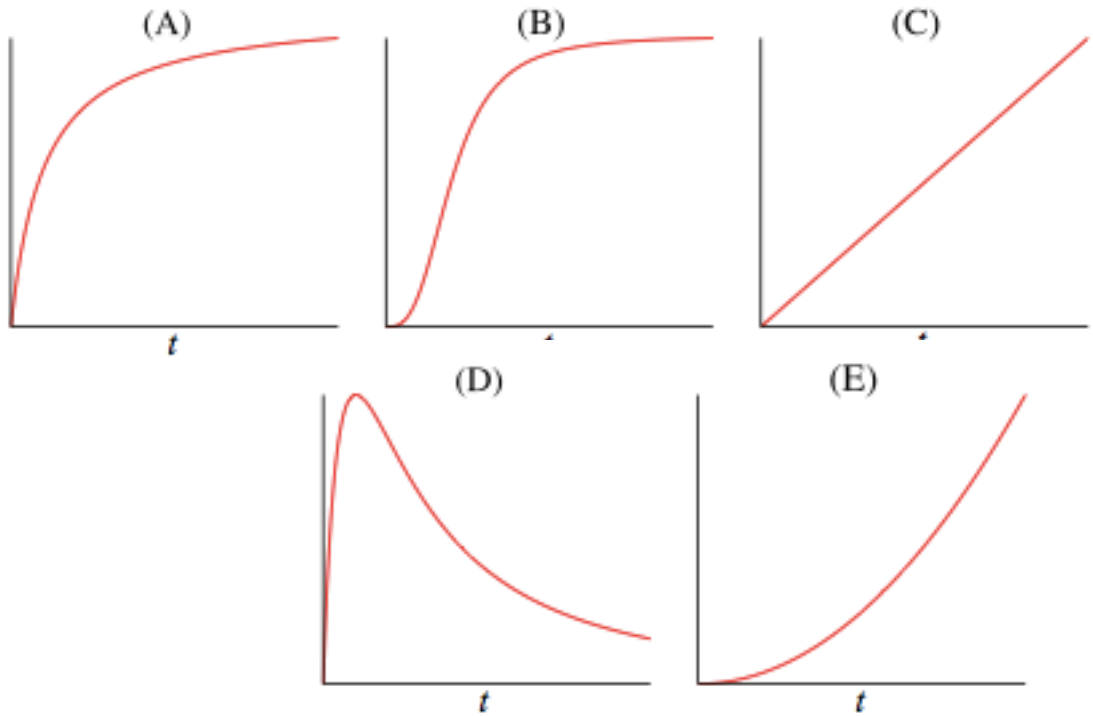
Ant trails

- 1 D
- 2 A
- 3 D
- 4 B
- 5 B
- 6 C
- 7 E
- 8 B
- 9 C
- 10 B

Problems to test your skill

MC 3

Each of the graphs represents f(t), the amount of food gained during time t spent in a food patch. Which of these graphs represents the type of patch where food can be collected rapidly at first, but there is only some maximal amount that can be obtained no matter how long the search continues?



MC 5

The absolute maximum of the function

$y = f(x) = x + 1/x$ on the interval $0.1 \le x \le 2$

occurs at

(a) $x = -1$

(b) $x = 0.1$

(c) $x = 1$

(d) $x = 2$

(e) $x = \pm 1$

MC 6

Let y = f(x) be a smooth function (derivatives of all orders exist) at x0. Which of the following statements is correct?

(a) If $f''(x_0) = 0$, then the function has an inflection point at x_0.

(b) If the function has an inflection point at x_0, then $f''(x_0) = 0$.

(c) Both (A) and (B) are correct.

(d) If $f''(x_0) = 0$, then the function has a critical point at x_0.

(e) If $f''(x_0) = 0$, then the function never has a critical point at x_0.

Related problems

Practice the Chain Rule

Find the derivatives of the following functions: Do not simplify.

(1) $f(x) = \dfrac{(x+1)^{10}}{(2x-1)^3}$

(2) $f(x) = (x^2-3x+1)^{-5}$

(3) $y = 4\pi r^2$, $\dfrac{dr}{dt} = 5$ find dy/dt

(4) $y = x^2+2$, $x = 1+\dfrac{1}{t}$ find dy/dt

Solution

Chain Rule etc.

(1) $f(x) = \dfrac{(x+1)^{10}}{(2x-1)^3}$, $f'(x) = \dfrac{\left[(x+1)^{10}\right]'(2x-1)^3 - \left[(2x-1)^3\right]'(x+1)^{10}}{\left[(2x-1)^3\right]^2}$

$$f'(x) = \frac{10(x+1)^9 \cdot 1 \cdot (2x-1)^3 - 3(2x-1)^2 \cdot 2 \cdot (x+1)^{10}}{(2x-1)^6}$$

(2) $f(x) = (x^2-3x+1)^{-5}$ $\qquad f'(x) = -5(x^2-3x+1)^{-5-1} \cdot (2x-3)$

$$= -5(x^2-3x+1)^{-6} \cdot (2x-3)$$

(3) $y = 4\pi r^2$, $\dfrac{dr}{dt} = 5 \Rightarrow \dfrac{dy}{dt} = \dfrac{dy}{dr}\dfrac{dr}{dt} = 8\pi r \cdot 5$

(4) $y = x^2+2$, $x = 1+\frac{1}{t} \Rightarrow \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = 2x \cdot \left[-\dfrac{1}{t^2}\right]$

$$= 2\left[1+\frac{1}{t}\right](-t^{-2})$$

•

Solutions to previous problems

First try out the problems at the end of the last lecture slides. Only then should you “peek” at the answers.

Optimal foraging

7.28. **Rate of net energy gain while foraging and traveling:** Animals spend energy in traveling and foraging. In some environments this energy loss is a significant portion of the energy budget. In such cases, it is customary to assume that to survive, an individual would optimize the rate of *net* energy gain, defined as

$$Q(t) = \frac{\text{Net energy gained}}{\text{total time spent}} = \frac{\text{Energy gained} - \text{Energy lost}}{\text{total time spent}} \tag{7.10}$$

Assume that the animal spends p energy units per unit time in all activities (including foraging and traveling). Assume that the energy gain in the patch ("patch energy function") is given by (7.6). Find the optimal patch time, that is the time at which $Q(t)$ is maximized in this scenario.

Solution

$$f(t) = \frac{E_{max}t}{k+t} \quad \text{where} \quad E_{max}, k > 0, \text{ are constants.} \tag{7.6}$$

Solution

The rate of *net* energy gain can be written as

$$Q(t) = \frac{\text{Energy gained} - \text{Energy lost}}{\text{total time spent}}.$$

If the animal forages for a time t, it gains $f(t)$ energy units, where f is the function in (7.6). At the same time, it spends an amount of energy $p(t+\tau)$ during its foraging and traveling. Then

$$Q(t) = \frac{f(t) - p(t+\tau)}{t+\tau} = \frac{f(t)}{t+\tau} - p = R(t) - p$$

Hence Q'(t)=R'(t) (since p is a constant). so that maximizing Q turns out to be the same as maximizing R as done in the lecture. So t= $\sqrt{k\tau}$,

More optimal foraging

7.29. **Maximizing net energy gain:** Suppose that the situation requires an animal to maximize its net energy gained $E(t)$ defined as

$$E(t) = \text{energy gained while foraging} - \text{energy spent while foraging and traveling.}$$

(This means that $E(t) = f(t) - r(t+\tau)$ where r is the rate of energy spent per unit time and τ is the fixed travel time).

Assume as before that the energy gained by foraging for a time t in the food patch is $f(t) = E_{max}t/(k+t)$.

Solution

(a) Find the amount of time t spent foraging that maximizes $E(t)$.

(b) Indicate a condition of the form $k < \boxed{?}$ that is required for existence of this critical point.

- **Solution to Problem 7.29:**

We find the derivative of $E(t)$ and set it to zero to get the critical point:

$$E(t) = \frac{E_{max}t}{k+t} - r(t+\tau) \quad \Rightarrow \quad E'(t) = E_{max}\frac{k}{(k+t)^2} - r = 0$$

So

$$E_{max}\frac{k}{(k+t)^2} = r, \quad \Rightarrow \quad (k+t)^2 = E_{max}\frac{k}{r}, \quad \Rightarrow \quad (k+t) = \sqrt{E_{max}\frac{k}{r}}$$

Hence we find that

$$t = \sqrt{E_{max}\frac{k}{r}} - k.$$

For this to exist we need $t > 0$, which implies $\sqrt{E_{max}\frac{k}{r}} > k$. We can rewrite this as $E_{max}\frac{k}{r} > k^2$ or $k < E_{max}/r$.

Cannon ball

. A cannonball is shot vertically upwards from the ground with initial velocity $v_0 = 15\text{m/sec}$. It is determined that the height of the ball, y (in meters), as a function of the time, t (in sec) is given by

$$y = v_0 t - 4.9t^2$$

Determine the following:

(a) The time at which the cannonball reaches its highest point,

(b) The velocity and acceleration of the cannonball at $t = 0.5$ s, and $t = 1.5$ s.

(c) The time at which the cannonball hits the ground.

Solution

Solution:

(a) The cannonball reaches its highest point when

$$v(t) = \frac{dy}{dt} = v_0 - 9.8t = 15 - 9.8t = 0,$$

that is when $t = \dfrac{15}{9.8} \approx 1.53$ sec.

(b) The velocity is $v(t) = v_0 - 9.8t = 15 - 9.8t$. Therefore the velocities at times 0.5 and 1.5 are $v(0.5) = 10.1$ m/sec and $v(1.5) = 0.3$ m/sec. The acceleration at any time t during the flight, including $t = 0.5,\ 1.5$, is $a(t) = -9.8$ m/sec^2.

(c) The cannonball hits the ground when $y = 0$:

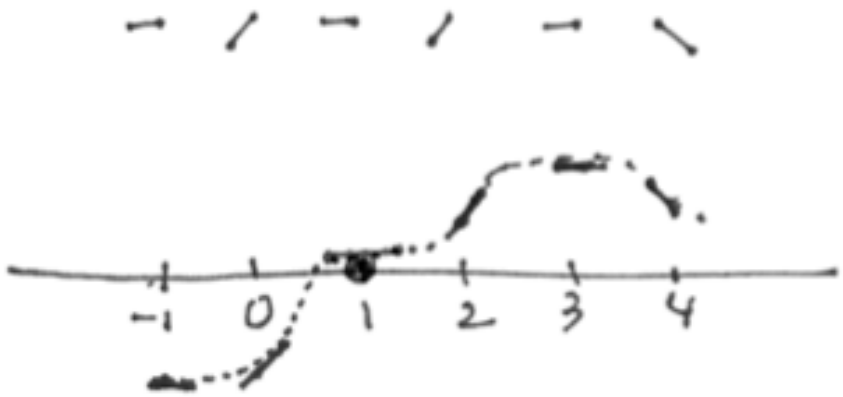
$$0 = 15t - 4.9t^2 \Longrightarrow t(15 - 4.9t) = 0 \Longrightarrow t = \frac{15}{4.9} \approx 3.06 \text{ sec.}$$

Review problem (1)

(A) Sketch $f(x)$ given the following information: $f(1) = 0$

x	-1	0	1	2	3	4
$f'(x)$	0	$+$	0	$+$	0	$-$

Solution:

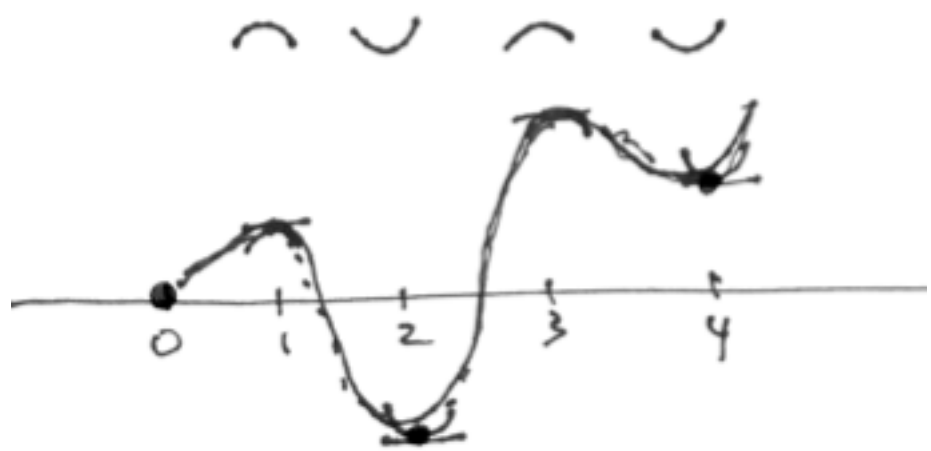


Review Problem (2)

(2) Sketch $f(x)$ given the following information

x	0	1	2	3	4
f	0		-1		1
f'		0	0	0	0
f''		$-$	$+$	$-$	$+$

Solution:



Review Problem (3)

(3) Consider the function $f(x) = x^3 - 3ax^2 + 3x$

(a) For what range of values of a does this function have no critical points?

(b) For what value of a does $f(x)$ have an inflection point at $x = 0$?

Solution

Solution to (3a)

(a) $f'(x) = 3x^2 - 6ax + 3 = 0$ ← C.P.

$$3(\underbrace{x^2 - 2ax + 1}) = 0$$

Q.F.

$$x = \frac{+2a \pm \sqrt{(2a)^2 - 4}}{2}$$

there is no real soln when

$$4a^2 - 4 < 0$$
$$4a^2 < 4$$
$$a^2 < 1$$
$$-1 < a < 1$$

over this interval → no C.P.

Solution to (3b)

(b) Val. of a such that IP at $x=0$

$$f''(x) = 6x - 6a = 6(x-a)$$

when $x=a$ there is an I.P.

since: (1) $f''(a)=0$

(2) " changes sign.

$$f'''(x) = 6 \neq 0$$

$x=a$ is IP so if $a=0$ then IP is at $x=0$.