Find the critical points

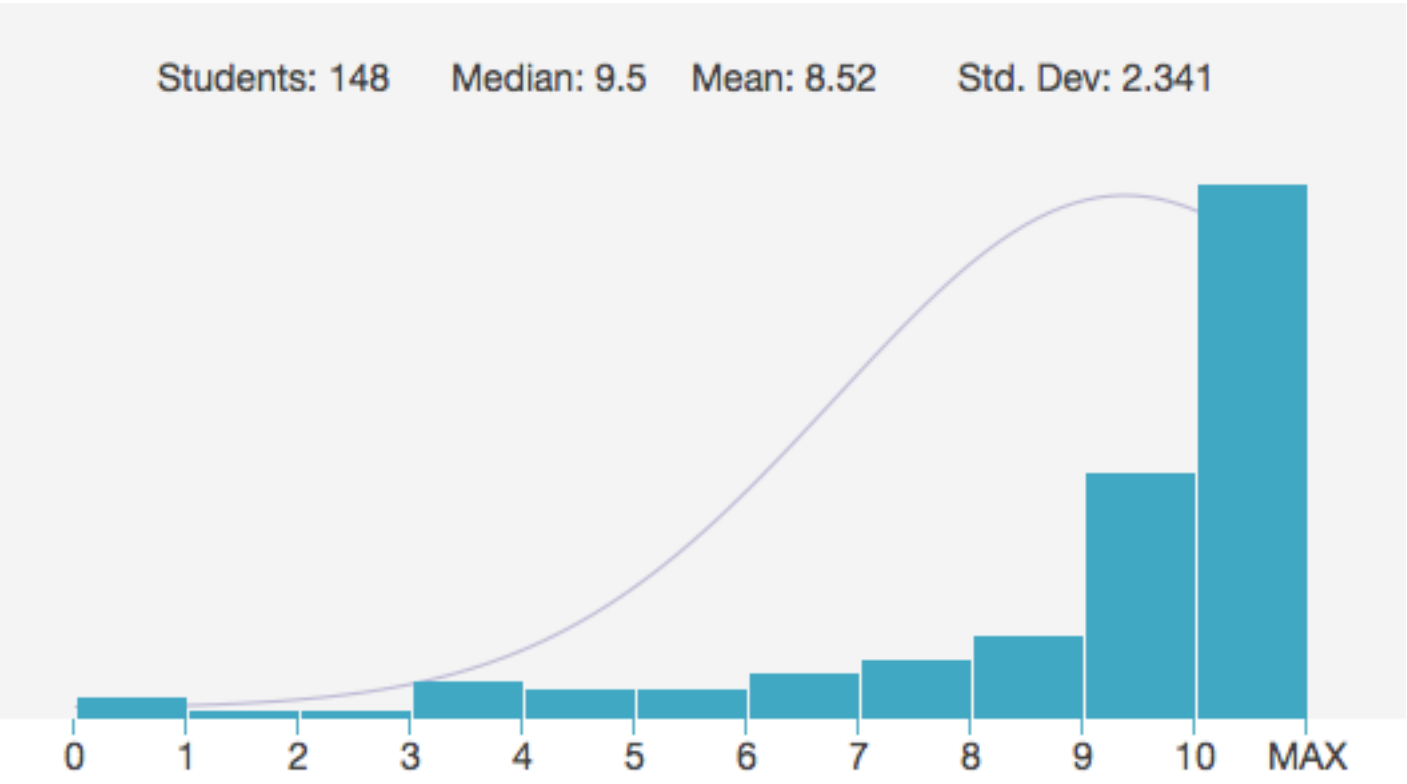
- (1) For the function $$R(t) = \frac{E_{max}t}{(k+t)(\tau+t)}$$

  τ, $E_{max}, k > 0$, *are constants.*

- (2) For $f(a) = (y_1 - ax_1)^2 + (y_2 - ax_2)^2$

  Where (x_i, y_i) are known <u>constants.</u>

Quiz2 scores

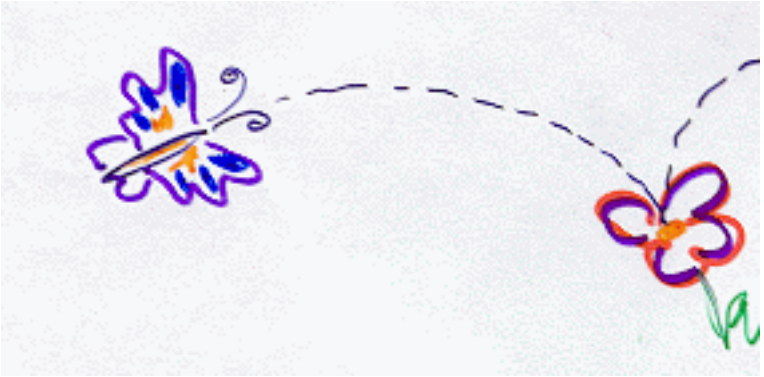
- 

Midterm

- Be sure you have signed up!
- Study every day between now and Oct 26! This is a great time to earn your points towards the test!
- Sample midterms (with solutions):

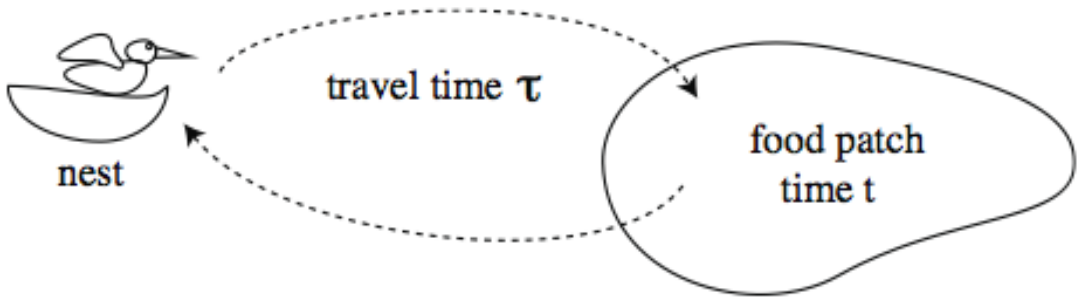
https://wiki.math.ubc.ca/mathbook/M102/Midterm_information

Optimal foraging



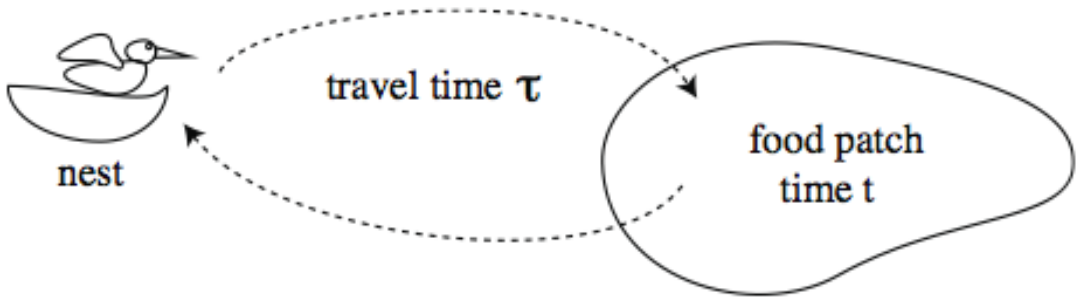
Foraging

- Time is limited.. There is lot's to do! How much time should be spent gathering food?



- Assume animal has to travel to a food patch, where it must spend time looking for food..

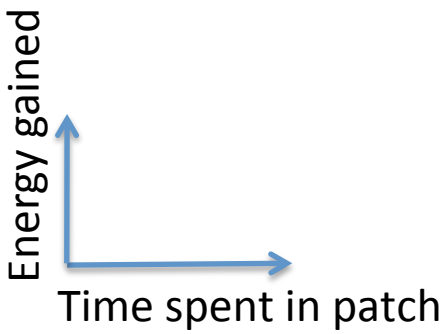
Definitions

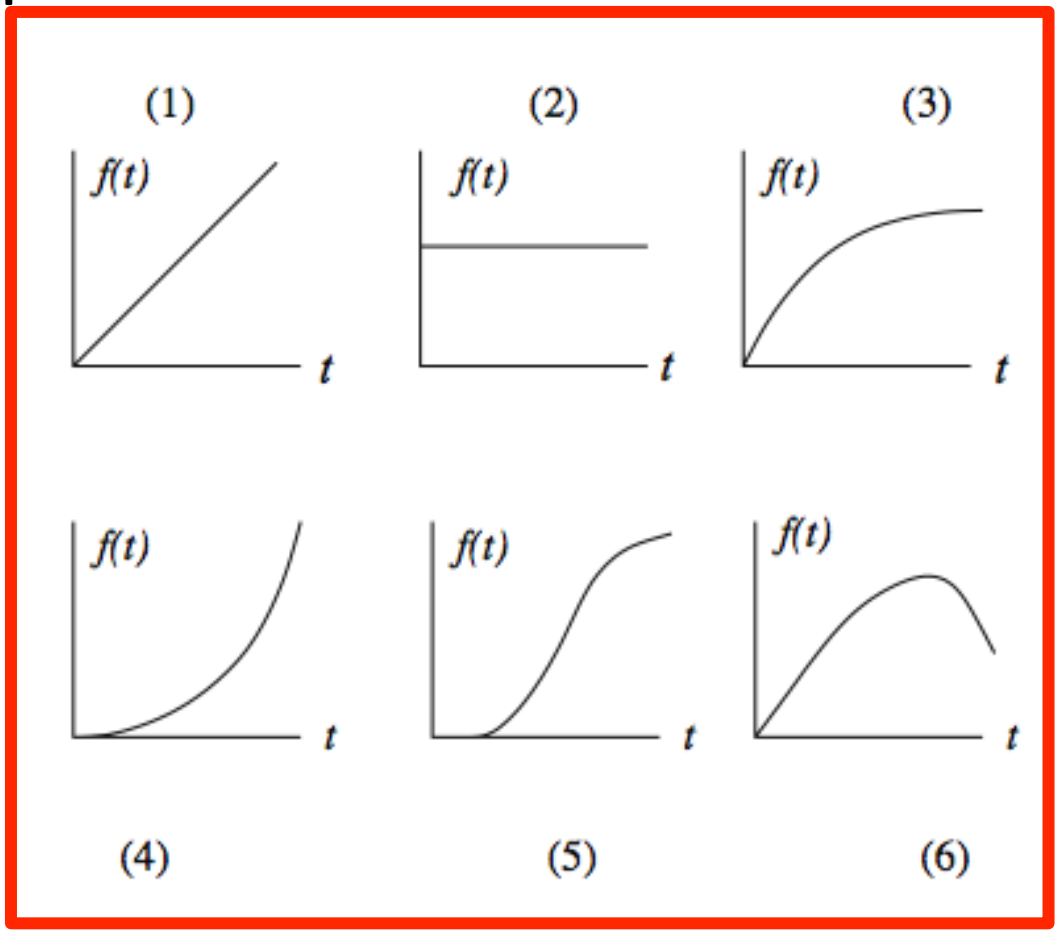


- τ= travel time between nest and **food patch**. (This is considered as time that is unavoidably wasted.) It is assumed to be a known constant value.
- t = **residence time** in the patch (i.e. how long to spend foraging in one patch), also called **foraging time**, This is a variable. We will try to find the optimal time!
- $f(t)$ = energy gained by foraging in a patch for time t,

Energy gained in food patch

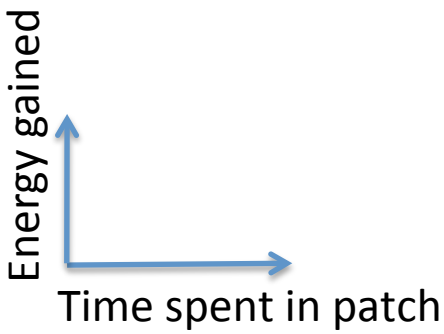
- Various types of patches

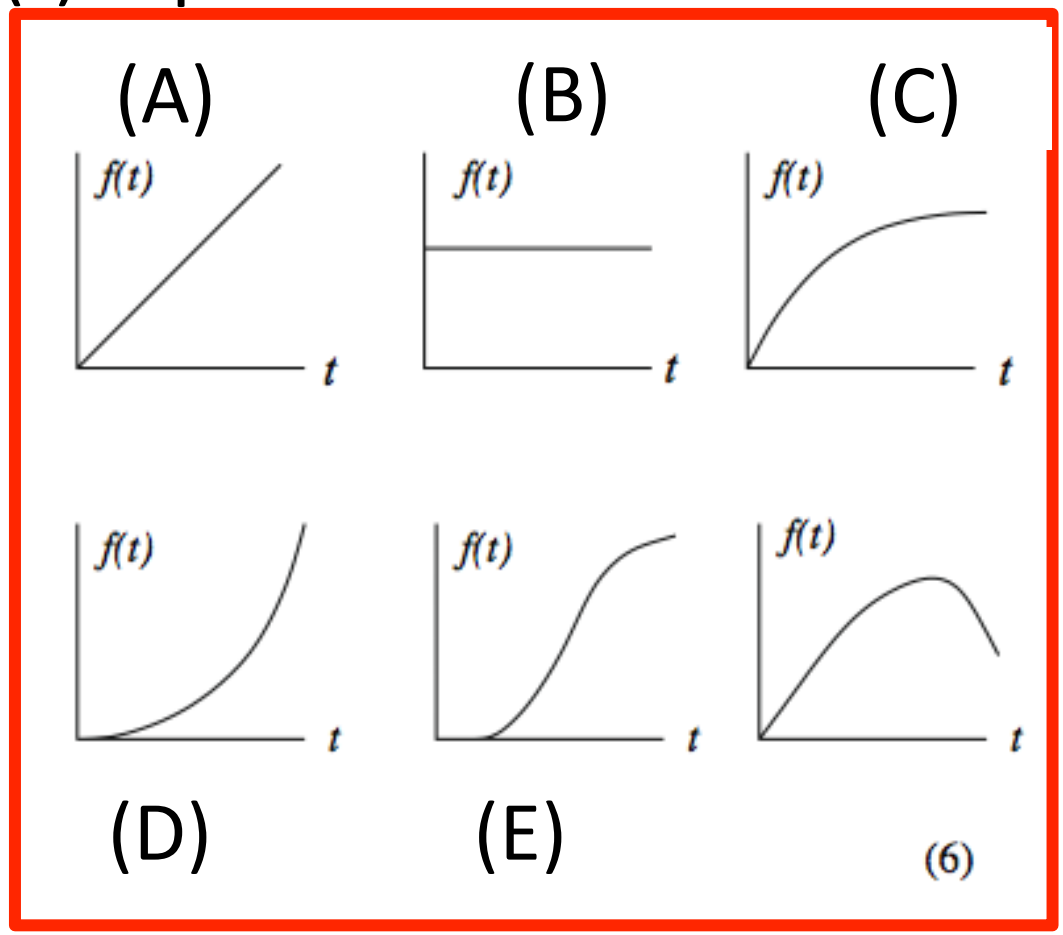




(1) The energy I gain is proportional to the time I spend in the food patch

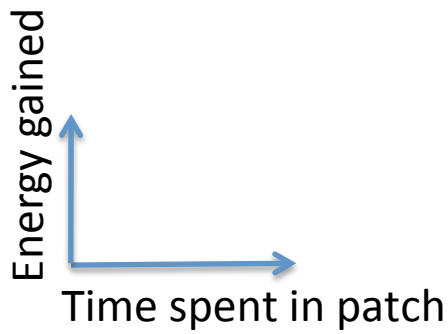
- Which function f(t) represents this?

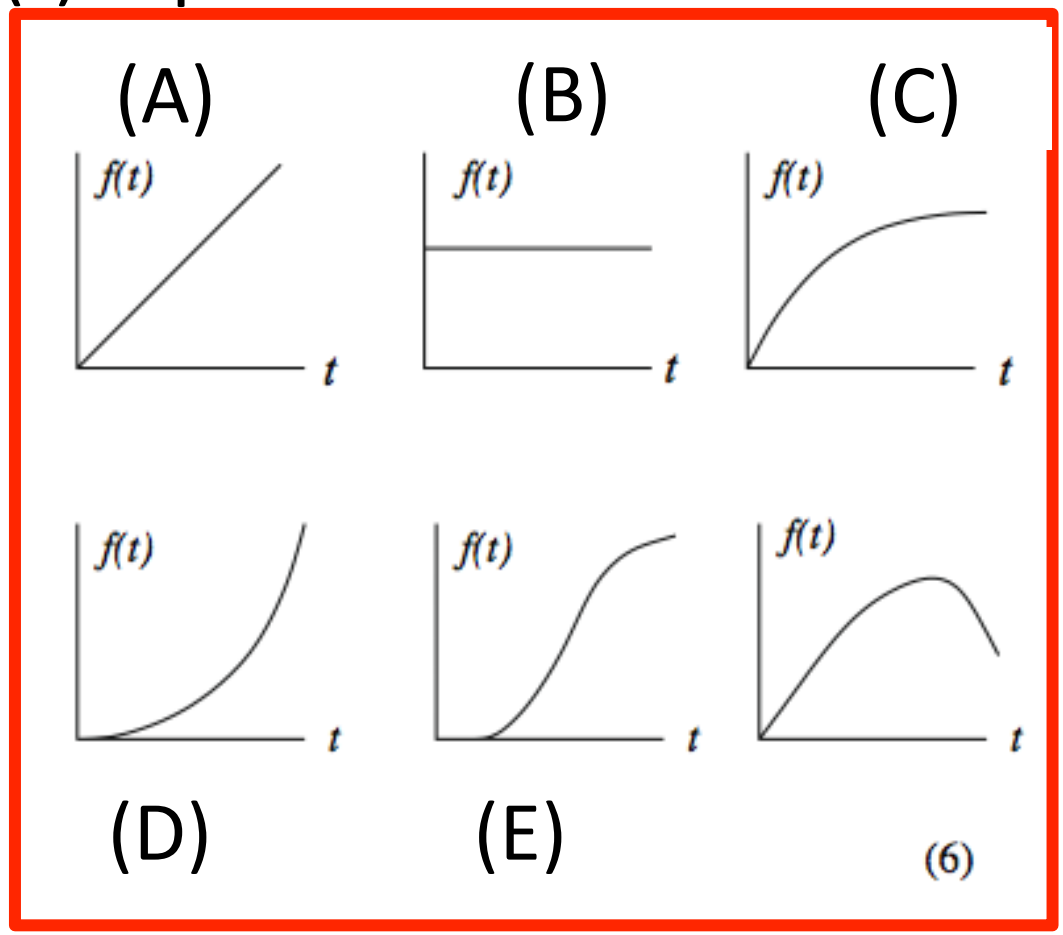




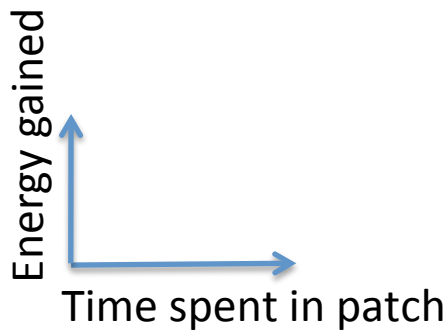
(2) There is some delay before I find food. After a while there is no more food left in the patch.

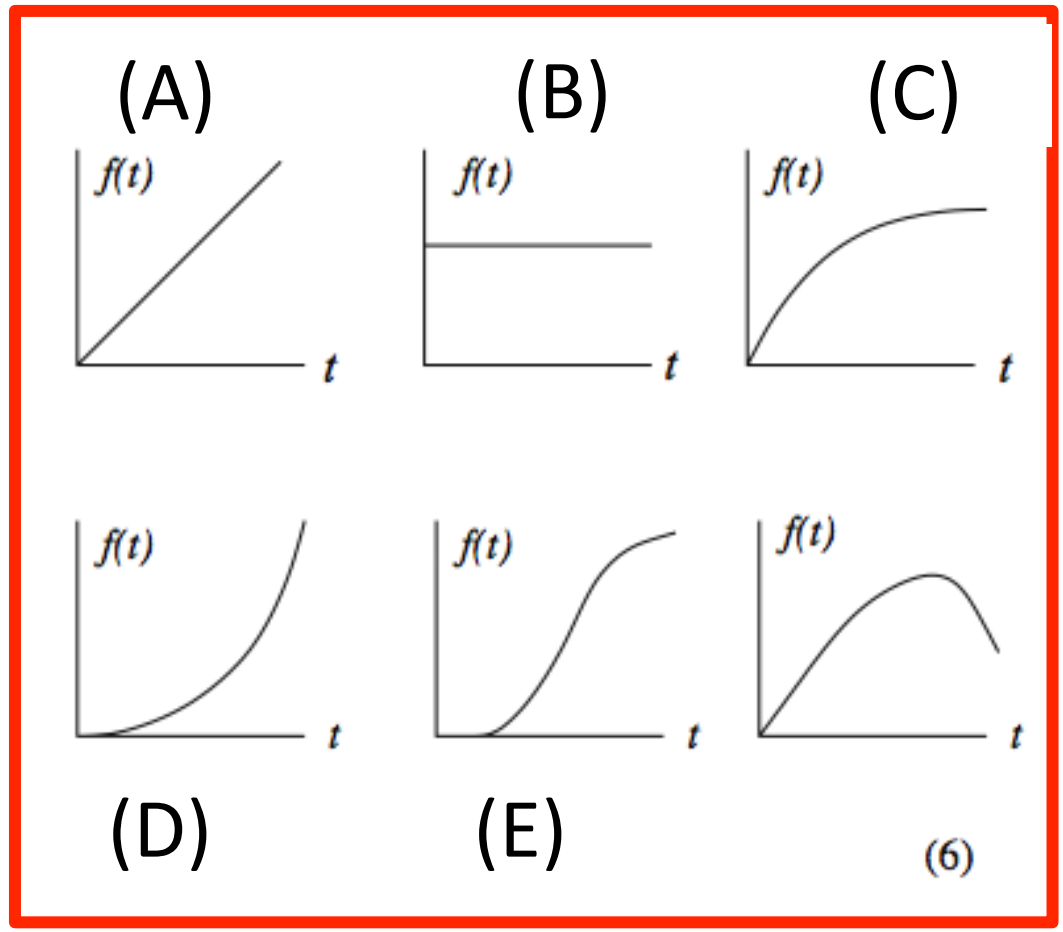
- Which function f(t) represents this?



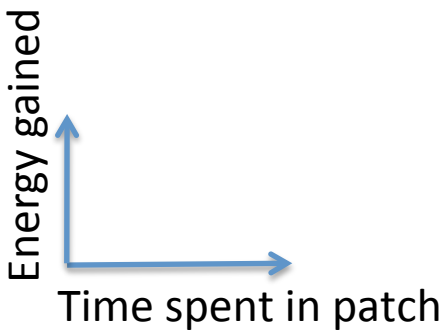


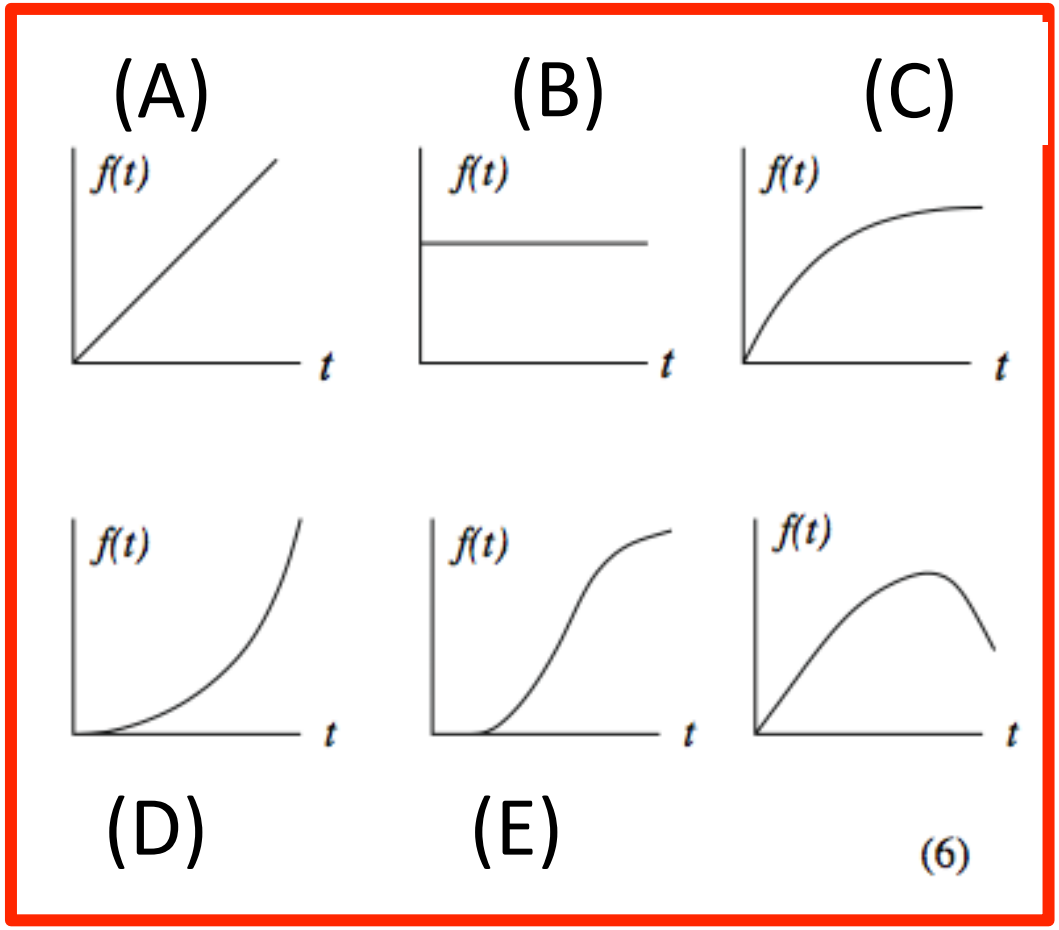
(3) The rate of energy gain keeps increasing with time





(4) At first I gain energy rapidly, but then I slow down as I get satisfied.





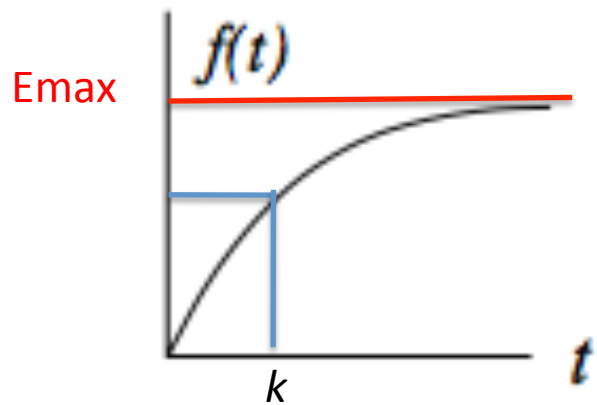
Being most efficient

We consider the case that the animal wants to get the most energy per unit time.

A specific example:

- Assume that

$$f(t) = \frac{E_{max} t}{k + t} \tag{3}$$



- Maximize the function

$$R(t) = \frac{f(t)}{\tau + t}.$$

From the video and notes:

Maximize the function $R(t) = \frac{f(t)}{\tau + t}$. for $f(t) = \frac{E_{max} t}{k + t}$

- Function to optimize: $R(t) = \frac{E_{max} t}{(k+t)(\tau+t)}$
- Set derivative to zero: $R'(t) = E_{max} \frac{k\tau - t^2}{(k+t)^2(\tau+t)^2} = 0$
- Solution (crit pts): $k\tau - t^2 = 0 \quad \Rightarrow \quad t_{1,2} = \pm\sqrt{k\tau}.$
- Keep the positive root: $t = \sqrt{k\tau}$

(5) Are we done?

$$t = \sqrt{k\tau}$$

- (A) Yes, we have found the optimal time.
- (B) No, we still have to compute R(t) for this time and check that it is larger than R(0).
- (C) No, we need to check if there is a constraint to satisfy
- (D) No, we have to check that we found a local maximum

Always check the type of critical point!!

- Check that R''(t)<0 for $t = \sqrt{k\tau}$ and conclude that spending that much time in food patch maximizes the energy gain per unit time!

(6) Interpret the result

$$t = \sqrt{k\tau}$$

If it takes me a long time to get to the patch then, to maximize the energy gain per unit time, I should

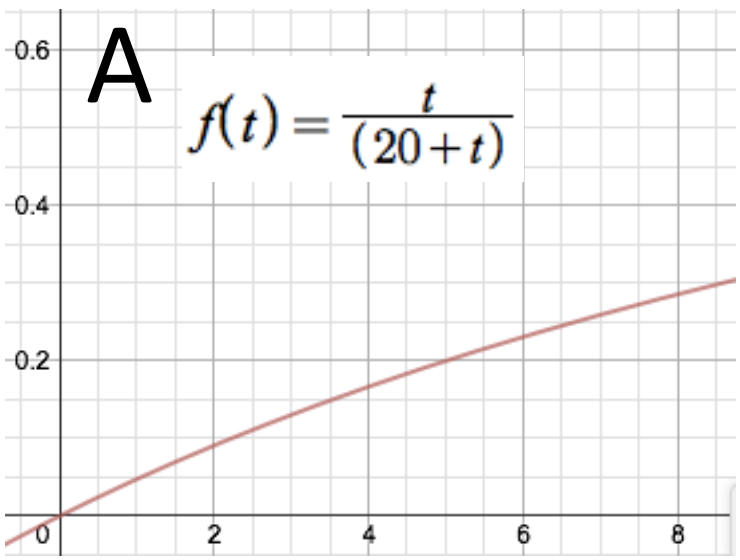
(A) Stay in the patch for a longer time

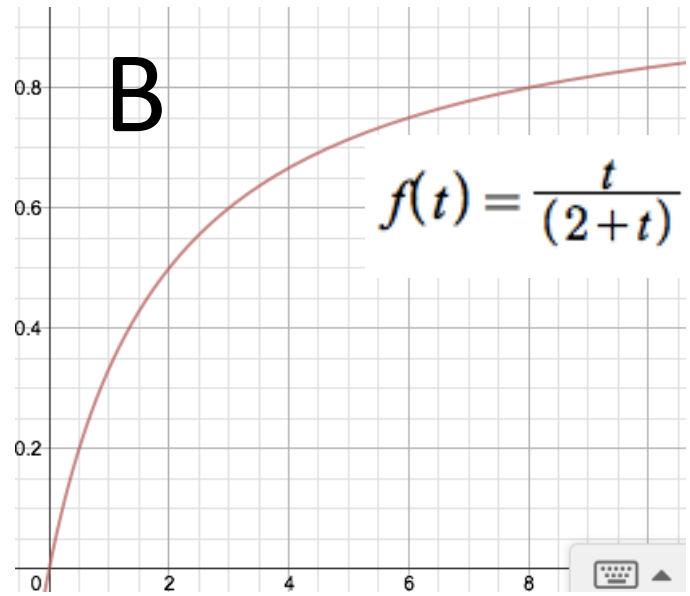
(B) Stay in the patch less time

(7) Interpret the result

$$t = \sqrt{k\tau}$$

Here are two types of patches. In which one of them should I spend more time in order to maximize energy gain per unit time?





Now a bit further!
Learning goal:

- To generalize the results
- To become more comfortable with abstract arguments
- To use geometric arguments

- (This will be a bit more subtle, but will stretch our minds!)

Goal: optimize energy per unit time

- How long to stay in food patch (find optimal t) to maximize:

$$R(t) = \frac{\text{Total energy gained}}{\text{total time spent}}$$

- Total time spent = travel time (wasted) + foraging time t

= τ + t

$$R(t) = \frac{f(t)}{\tau + t}.$$

$$R(t) = \frac{f(t)}{\tau + t}.$$

(8) The derivative of R(t) is:

- (A) R'(t)= $\frac{f'(t)}{\tau}$
- (B) R'(t)= $\frac{f'(t)}{(\tau + t)^2}$
- (C) R'(t)= $\frac{f'(t)(\tau + t) - f(t)}{(\tau + t)^2}$
- (D) R'(t)= $\frac{f'(t) - f(t)}{(\tau + t)^2}$
- (E) R'(t)= $f'(t)$

$$R'(t) = \frac{f'(t)(\tau + t) - f(t)}{(\tau + t)^2}$$

(9) Critical points of R(t) satisfy R'(t)=0 so

- (A) $t = \sqrt{k\tau}$
- (B) $t_{1,2} = \pm\sqrt{k\tau}.$
- (C) $f'(t) = \frac{f(t)}{\tau + t}.$
- (D) $f'(t) = 0, f(t) = 0$
- (E) $f(t)(\tau + t) = f'(t)$

The optimal time (to maximize R) is the time at which

$$R'(t) = \frac{f'(t)(\tau + t) - f(t)}{(\tau + t)^2} = 0$$

Which is the same as the time at which

$$f'(t) = \frac{f(t)}{\tau + t}.$$

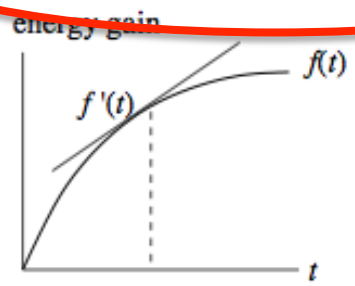
The optimal time (to maximize R) is the time at which

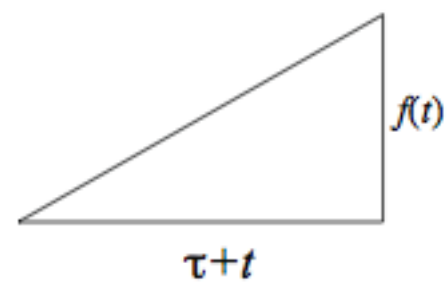
$$R'(t) = \frac{f'(t)(\tau + t) - f(t)}{(\tau + t)^2} = 0$$

Which is the same as the time at which

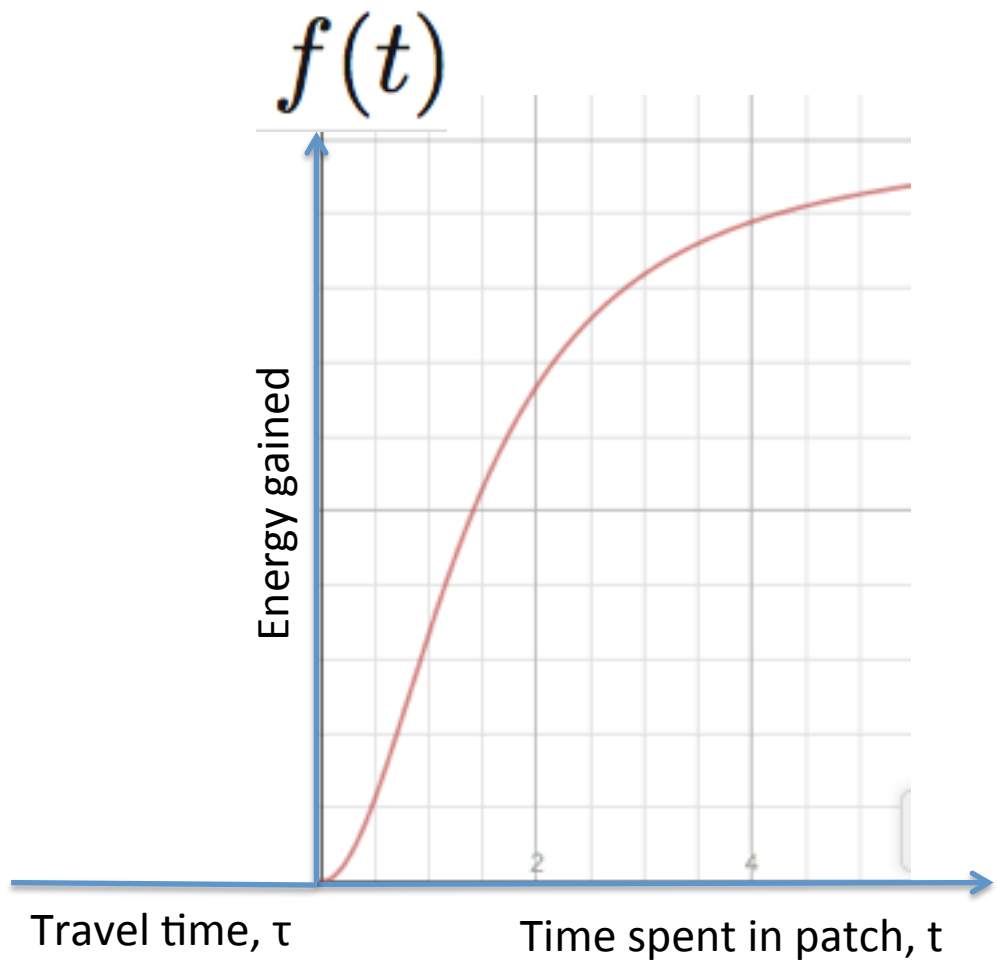
$$f'(t) = \frac{f(t)}{\tau + t}.$$

Slope of tangent line = ratio $\frac{f(t)}{\tau + t}$.



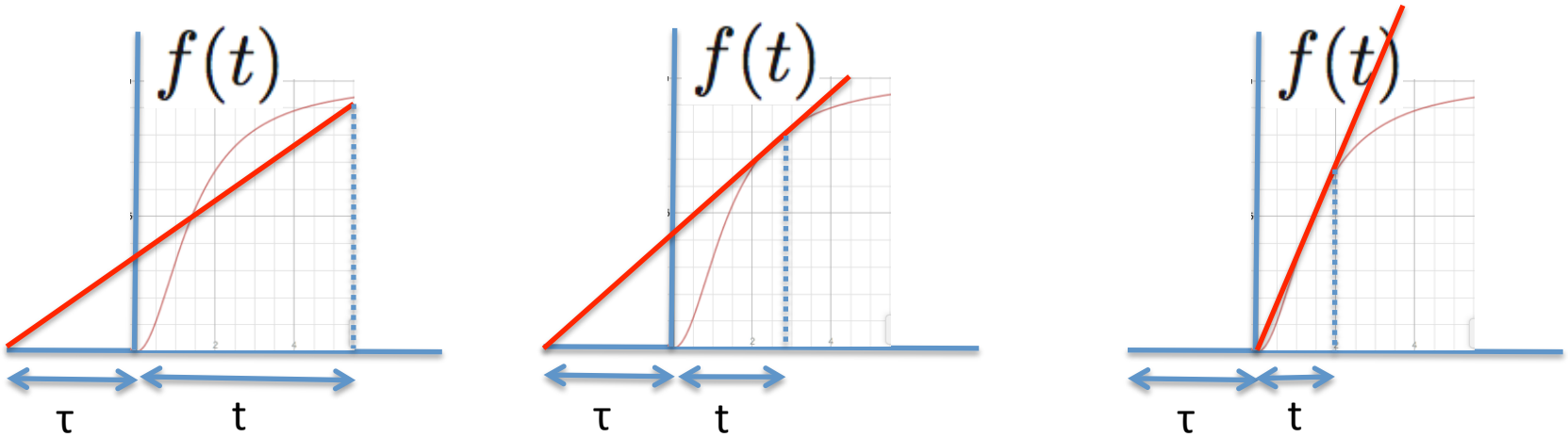


Suppose the energy gain function f(t) looks like this:



(10) In which of these diagrams is this equation satisfied?

$$f'(t) = \frac{f(t)}{\tau + t}.$$

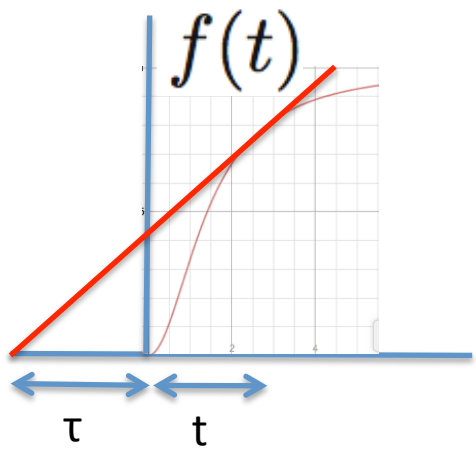


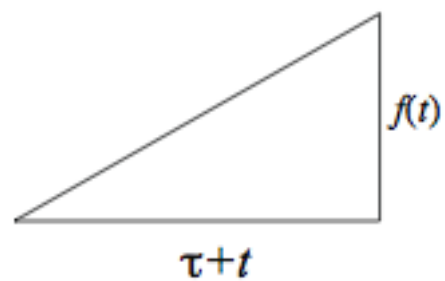
(A) (B) (C)

Conclusion:

- Construct a tangent line to the curve starting at the point (-τ,0).

$$f'(t) = \frac{f(t)}{\tau + t}.$$

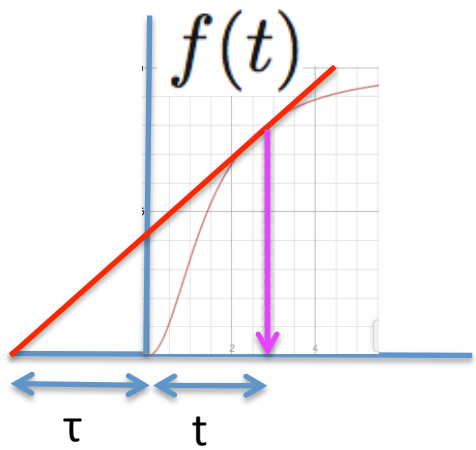


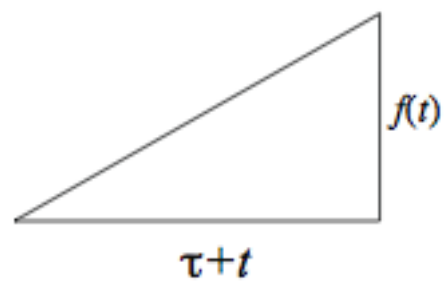


Conclusion:

- Construct a tangent line to the curve starting at the point (-τ,0). Find the value of t at which this line touches the curve.

$$f'(t) = \frac{f(t)}{\tau + t}.$$

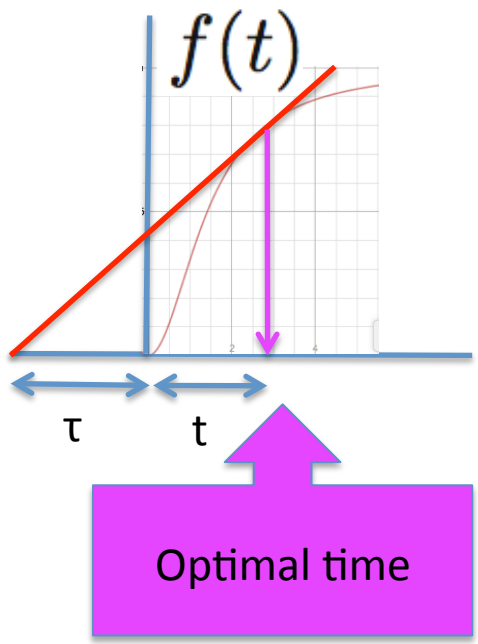


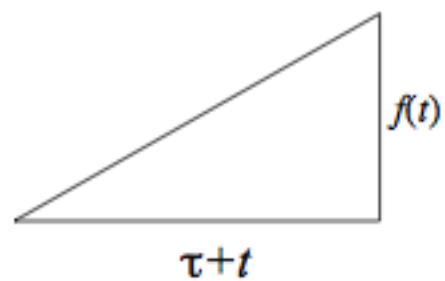


Conclusion:

- Construct a tangent line to the curve starting at the point (-τ,0). Find the value of t at which this line touches the curve. That's your optimal time!

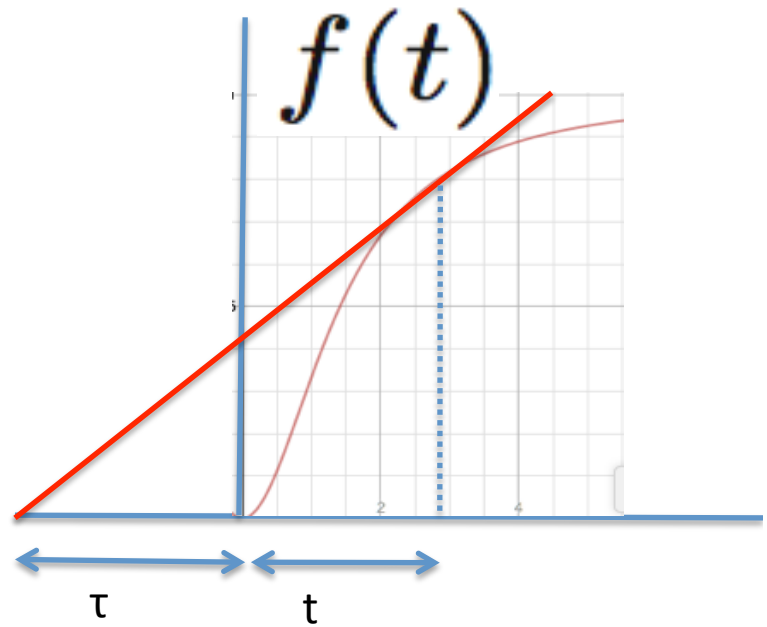
$$f'(t) = \frac{f(t)}{\tau + t}.$$





“Rooted tangent”

- This is a famous diagram in the Animal behaviour literature!
- It is a geometric solution.



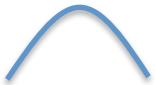
- It is possible (but not essential) to use similar reasoning in OSH 4 #1e (where tau=0 and you want a min, not a max)

Appendix
G.5
P 402

Remark:

- We have to verify that the time we found (geometrically) by this process leads to a maximum (not minimum) R(t) value.

- Exercise: Show that $R''(t) = \dfrac{f''(t)}{\tau + t}$

- If f is  then so is R
- If f is  then so is R

(11) Did we find an optimum?

$$R''(t) = \frac{f''(t)}{\tau + t}$$

Given that the second derivative of R(t) is as shown, we can conclude that

(A) There will be an optimal patch time that maximizes R(t) only if f(t) is a concave up.

(B) If f(t) is concave down then there is an optimal patch time that maximizes R(t).

(C) I don't get it.

Optimization used for fitting a trend-line to data

"Minimizing least squares of residuals"

(Linear regression)

UBC Math 102

Where can I find this material?

- M102 wiki

Page

MATH 102

Home
Announcements
Course calendar
Course notes
WeBWork
OSH
Pre-lecture videos
Other videos and demos
Spreadsheet help

Supplements

- Earth's energy balance
- Fitting data - least squares
- Optimal foraging and other repeated processes
- Numerical integration

Fitting a line to data (part 1 - $y = ax$)

Fitting a line to data (part 2 - $y = ax + b$)

Optimization applied to data fitting.

- We are given some data and want to describe its trend.
- How do we fit the best line through the data?

- "Least squares fitting" – a useful (and mathematically simple) procedure

Learning goals

- Understand what data fitting means in the simplest (linear least squares) setting.
- Understand the connection to optimization
- Be able to fit a line y=ax to 3 or more data points

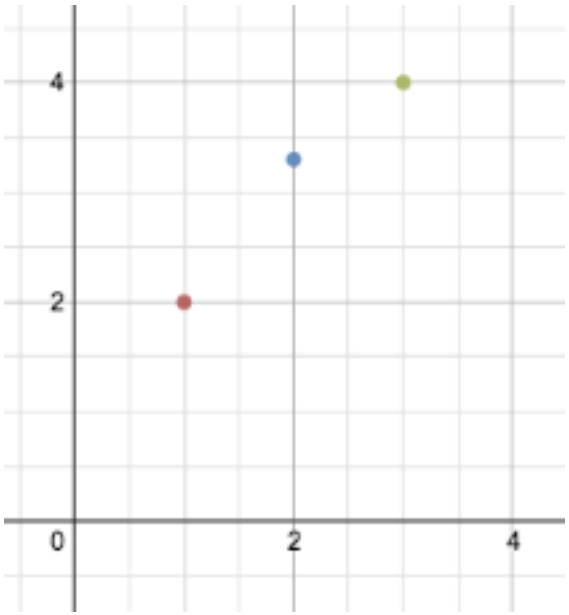
Example:

- Rain fall over three days

Day	1	2	3
Rain (cm)	2	3.3	4

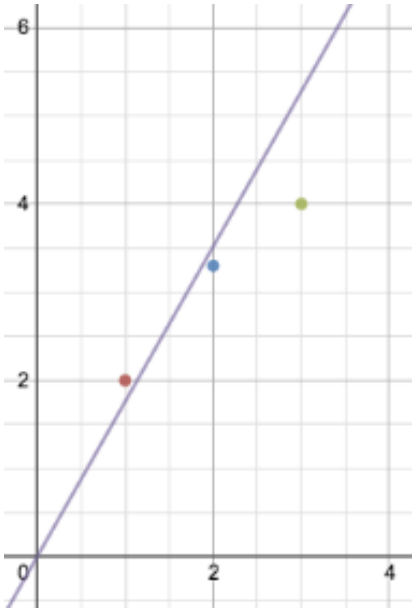
- Can we describe this by

$$y = ax$$

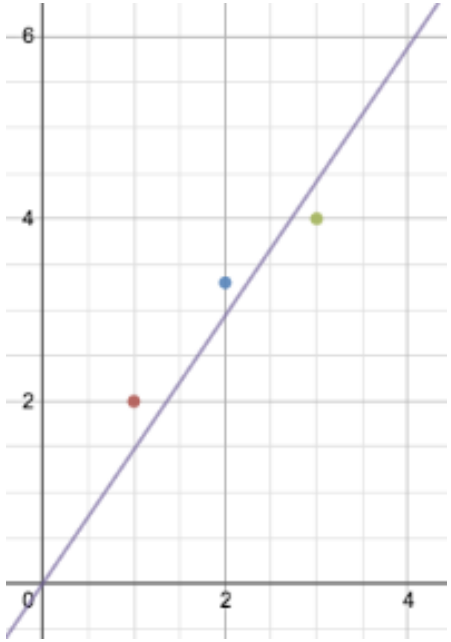


$y = ax$

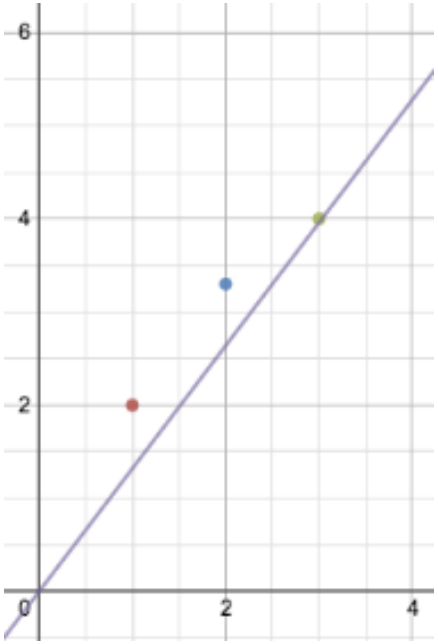
(1) Which line is "best fit"?



(A)



(B)



(C)

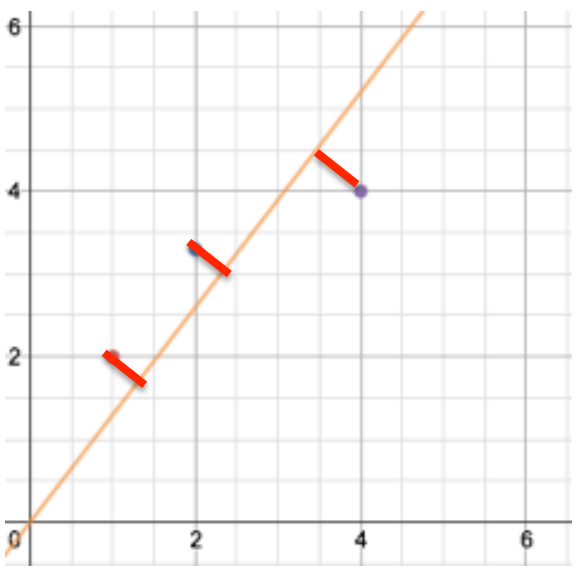
Can we come up with a formal process to get fit?

$y = ax$

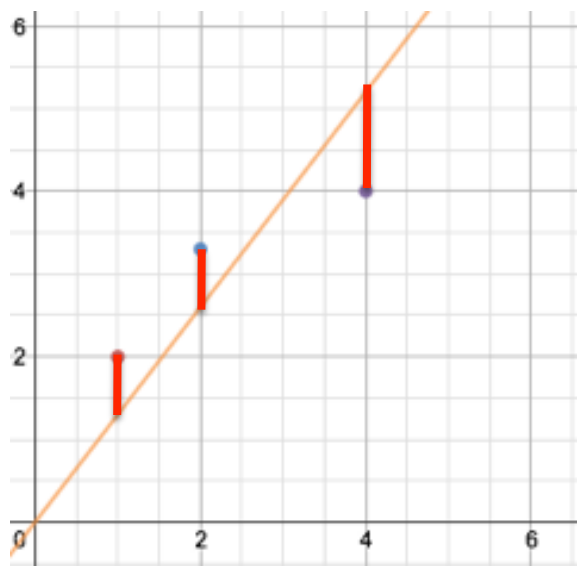
(2) We will define residuals

- These represent how far away the ideal line is from the actual data.
- According to the video, the residuals are:

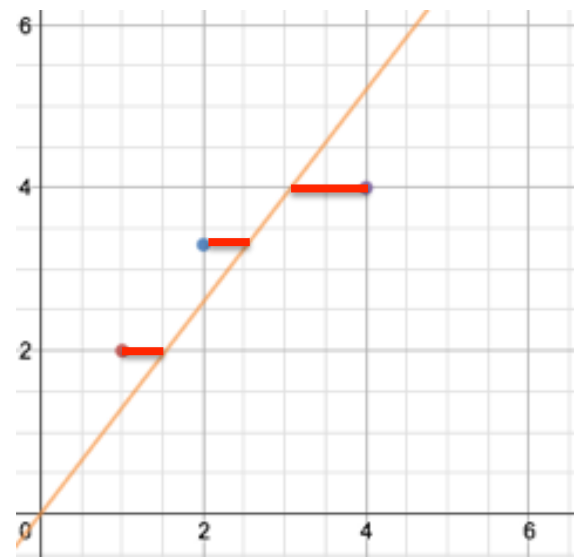
(A)



(B)



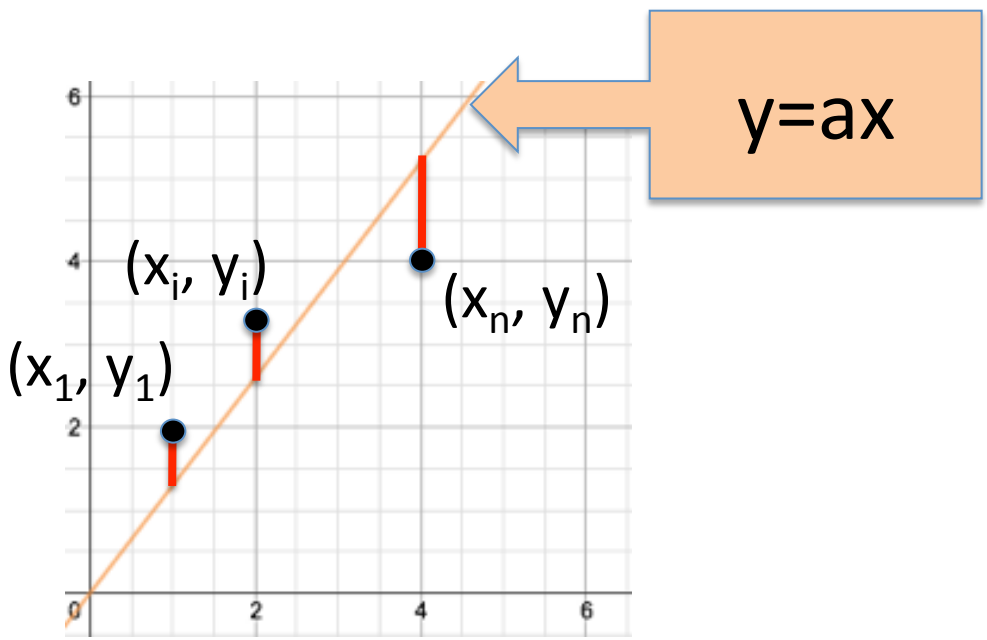
(C)



$y = ax$

Goal

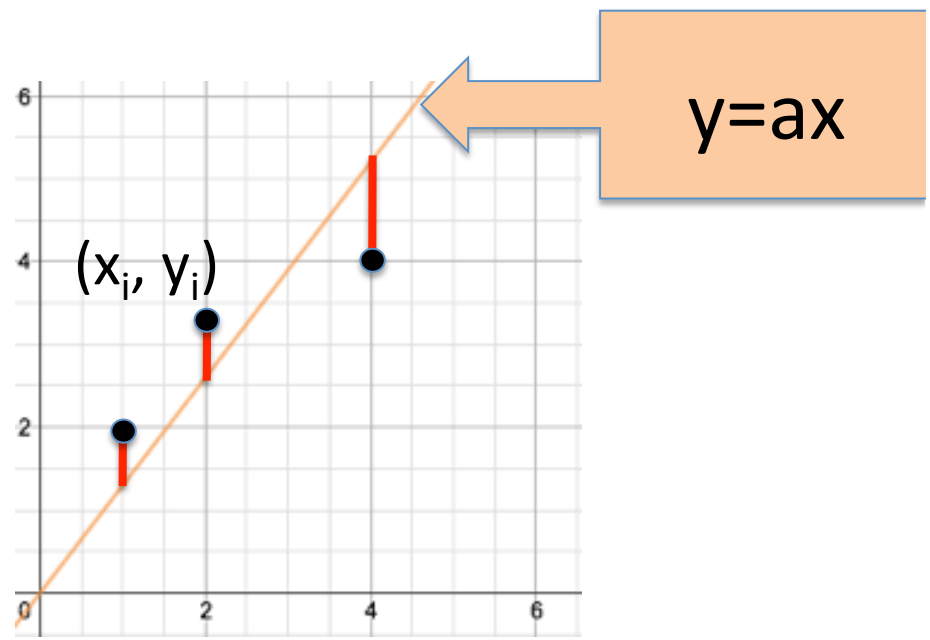
- Chose line for which the residuals are as small as possible! (minimize!)



$y = ax$

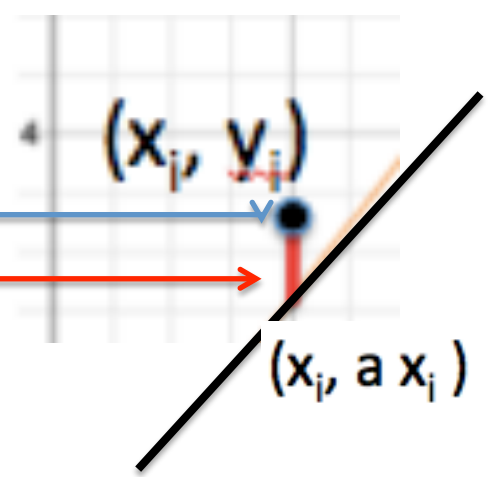
(3) A residual for (x_i, y_i) is:

- (A) $r_i = y_i^2 + x_i^2$
- (B) $r_i = a^2 (y_i^2 + x_i^2)$
- (C) $r_i = y_i - a\, x_i$
- (D) $r_i = y_i - x_i$
- (E) $r_i = x_i - y_i$



Residual

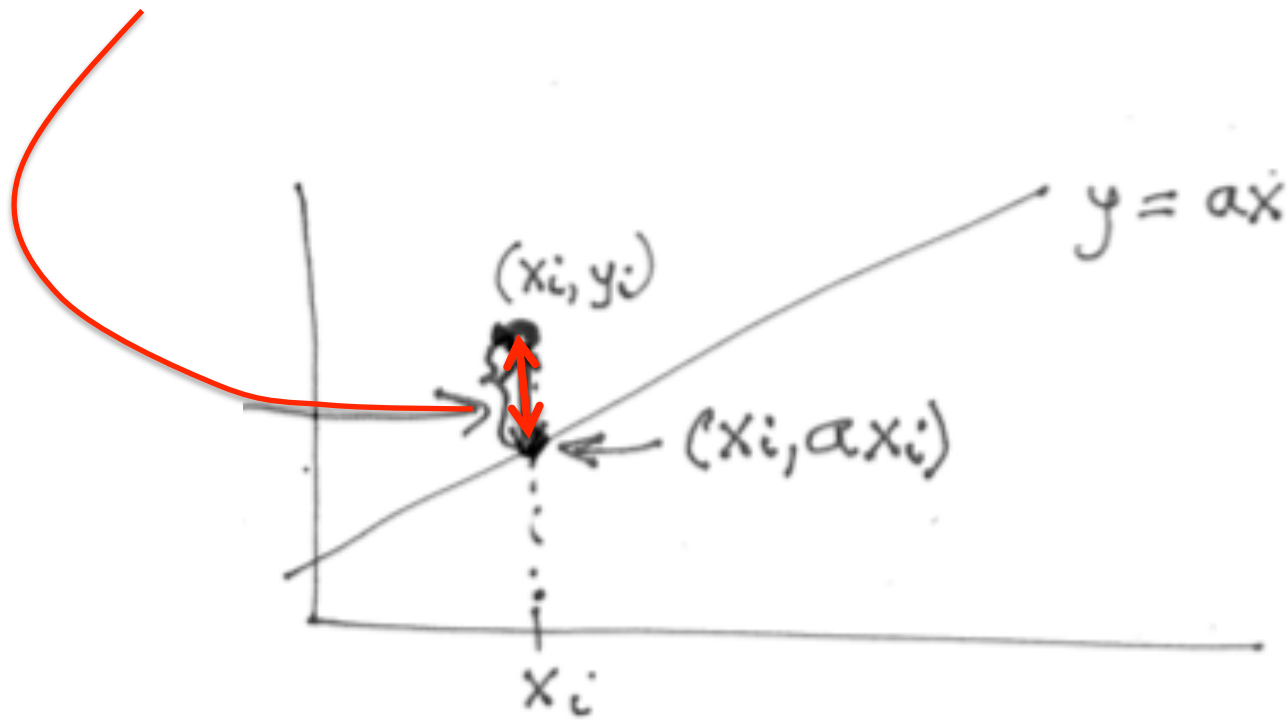
- If the data point was exactly ON THE LINE, then $y_i = a\,x_i$ which is same as $y_i - a\,x_i = 0$, so the residual is 0.
- We define the residual of a data point as the value of $y_i - a\,x_i$. This represents how far away the data point is from the line.
- Residual = $(y_i - a\,x_i)$
-



$y = ax$

Residual

- For the line y=ax and any data point (x_i, y_i), the residual is $(y_i - a\,x_i)$



$y = ax$

(4) Compute the residuals

Day x_i	1	2	3
Rain (cm) y_i	2	3.3	4

- (A)

residual	2-1	3.3-2	4-3

- (B)

residual	2-a	3.3-a	4-a

- (C)

residual	2-a	3.3-2a	4-3a

(5) To get the best fit line y=ax we should minimize:

- (A) $f(a) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_n - y_n)^2$
- (B) $f(a) = (y_1 - ax_1)^2 + (y_2 - ax_2)^2 + .. + (y_n - ax_n)^2$
- (C) $f(a) = |y_1 - ax_1| + |y_2 - ax_2| + .. + |y_n - ax_n|$
- (D) $f(a) = (y_1 - ax_1)(y_2 - ax_2)..(y_n - ax_n)$
- (E) $f(a) = (y_1 - ax_1) + (y_2 - ax_2) + .. + (y_n - ax_n)$

$y = ax$

"Sum of square residuals"

- SSR= $(y_1-ax_1)^2 + (y_2-ax_2)^2 + \ldots + (y_n-ax_n)^2$
- Short-hand notation:

  SSR(a) = $\sum_{i=1}^{N} (y_i - ax_i)^2$

Minimizing SSR(a) is equivalent to finding the slope of the line for which the deviations of data from the line are smallest overall.

Compute the squares of residuals

Day x_i	1	2	3
Rain (cm) y_i	2	3.3	4
residual	2-a	3.3-2a	4-3a
$(\text{residual})^2$	$(2-a)^2$	$(3.3-2a)^2$	$(4-3a)^2$

- $SSR=f(a) = (2-a)^2 + (3.3-2a)^2 + (4-3a)^2$

We want to minimize this!

Find $f'(a)$, set $f'(a)=0$, solve for a !

"Least squares"

Chain-rule says that

- The derivative of $f(a) = (5 - 4a)^2$

is

$$f'(a) = 2(5 - 4a) \cdot (-4)$$

Find the derivative of the SSR with resepct to *a*

f(*a*) = $(2-a)^2 + (3.3-2a)^2 + (4-3a)^2$

(6) Find the derivative of

$$f(a) = (2-a)^2 + (3.3-2a)^2 + (4-3a)^2$$

(A) f'(a)= 2(-20.6+14 a)

(B) f'(a)= -20.6+14 a

(C) f'(a)= 2(20.6-14 a)

(D) f'(a)= 2(9.3-6a)

(E) f'(a)= 30.89-41.2a+14a^2

(7) OK, so what is the best fit *a*?

Find the value of a that minimizes f(*a*)

(A) *a*= 1.15

(B) *a*= 1.25

(C) *a*= 1.47

(D) *a*= 2.11

OK, so what is the best fit *a*?

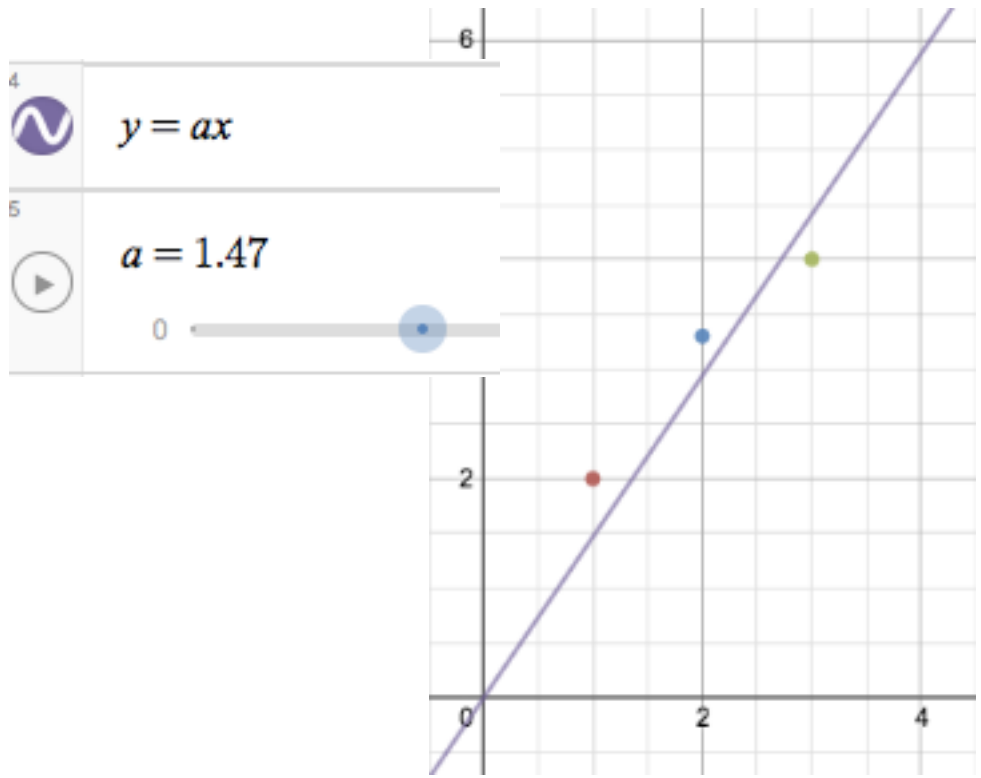
Find the value of a that minimizes f(*a*)

(A) *a*= 1.15

(B) *a*= 1.25

(C) *a*= 1.47

(D) *a*= 2.11



Observations

- Where did that come from and what pattern can we observe from this calculation???

Find the derivative of

$$f(a) = (2-a)^2 + (3.3-2a)^2 + (4-3a)^2$$

$$f'(a) = 2\cdot(2-a)\cdot(-1) + 2\cdot(3.3-2a)\cdot(-2) + 2\cdot(4-3a)\cdot(-3)$$

Take out factor of 2, simplify and group terms

$$f'(a) = -2\left[(2\cdot 1 - a\cdot 1^2) + (3.3\cdot 2 - a\cdot 2^2) + (4\cdot 3 - a\cdot 3^2)\right]$$

$$f'(a) = -2\left[(2\cdot 1 + 3.3\cdot 2 + 4\cdot 3) - a(1^2 + 2^2 + 3^2)\right]$$

Find the critical point

$$f'(a) = -2\left[(2\cdot 1 + 3.3\cdot 2 + 4\cdot 3) - a(1^2 + 2^2 + 3^2)\right]$$

$$f'(a) = 0$$

$$a = \frac{(2\cdot 1 + 3.3\cdot 2 + 4\cdot 3)}{(1^2 + 2^2 + 3^2)}$$

$$a = \frac{20.6}{14} = 1.47$$

OK, so what is the best fit *a*?

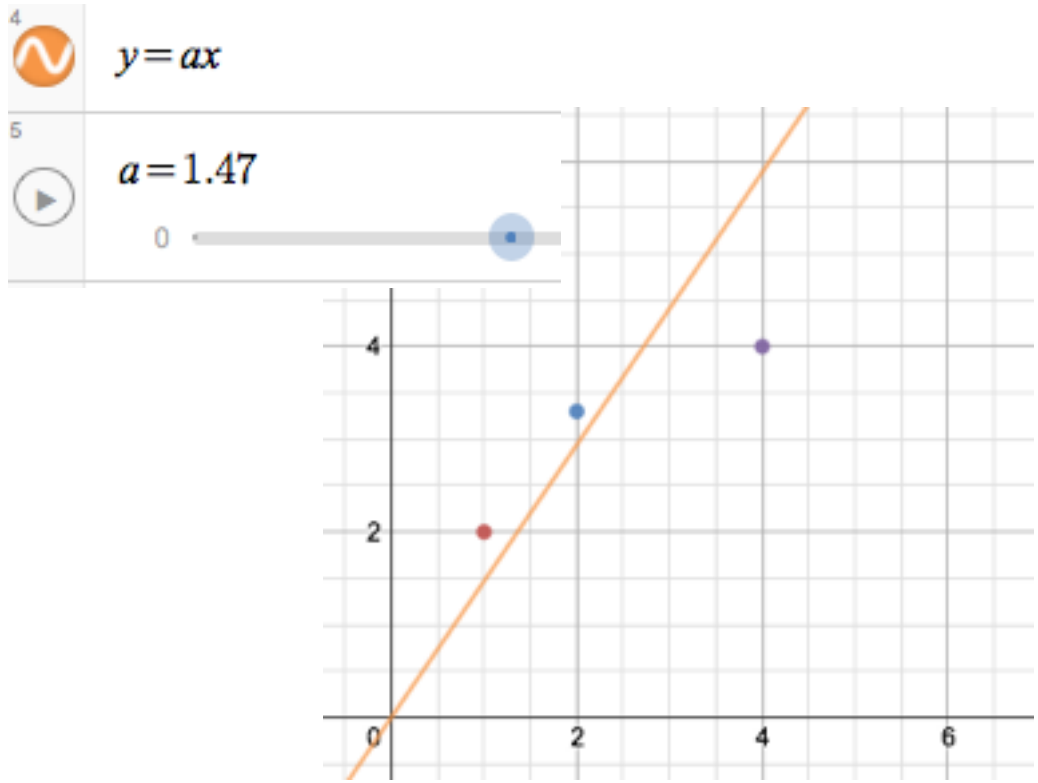
Find the value of a that minimizes f(*a*)

(A) *a*= 1.15

(B) *a*= 1.25

(C) *a*= 1.47

(D) *a*= 2.11



More general case:

- SSR= $f(a) = \sum_{i=1}^{N} (y_i - ax_i)^2$

$$f(a) = (y_1 - ax_1)^2 + \ldots (y_i - ax_i)^2 + (y_N - ax_N)^2$$

- Find *a* that minimizes this!
- Note that the x's and y's are known data values!!

More general case:

- SSR= $f(a) = \sum_{i=1}^{N} (y_i - ax_i)^2$

$$f(a) = (y_1 - ax_1)^2 + \ldots \boxed{(y_i - ax_i)^2} + (y_N - ax_N)^2$$

- Derivative:

$$f'(a) = .. + \ldots 2(y_i - ax_i) \cdot (-x_i) + \ldots$$

$$f'(a) = \ldots - 2(y_i x_i - a x_i x_i) + \ldots$$

More general case:

- SSR= $f(a) = \sum_{i=1}^{N} (y_i - ax_i)^2$

$$f(a) = (y_1 - ax_1)^2 + \ldots (y_i - ax_i)^2 + (y_N - ax_N)^2$$

- Derivative $f'(a) = -2 \sum (y_i x_i - a x_i x_i)$

$$f'(a) = -2 \left(\sum y_i x_i - a \sum x_i^2 \right) = 0$$

(critical point)

The desired value of *a*:

$$f'(a) = -2\left(\sum y_i x_i - a\sum x_i^2\right) = 0$$

$$a = \frac{\sum y_i x_i}{\sum x_i^2}$$

- The optimal value of the slope of the line to fit the data!

Answers

FORAGING

- 1 A
- 2 E
- 3 D
- 4 C
- 5 D
- 6 A
- 7 A
- 8 C
- 9 C
- 10 B
- 11 B

Data fitting

- 1 B
- 2 B
- 3 C
- 4 C
- 5 B
- 6 A
- 7 C

Problems to test your skill

Optimal foraging

7.28. **Rate of net energy gain while foraging and traveling:** Animals spend energy in traveling and foraging. In some environments this energy loss is a significant portion of the energy budget. In such cases, it is customary to assume that to survive, an individual would optimize the rate of *net* energy gain, defined as

$$Q(t) = \frac{\text{Net energy gained}}{\text{total time spent}} = \frac{\text{Energy gained} - \text{Energy lost}}{\text{total time spent}} \tag{7.10}$$

Assume that the animal spends p energy units per unit time in all activities (including foraging and traveling). Assume that the energy gain in the patch ("patch energy function") is given by (7.6). Find the optimal patch time, that is the time at which $Q(t)$ is maximized in this scenario.

$$f(t) = \frac{E_{max}t}{k+t} \quad \text{where} \quad E_{max}, k > 0, \text{ are constants.} \tag{7.6}$$

More optimal foraging

7.29. **Maximizing net energy gain:** Suppose that the situation requires an animal to maximize its net energy gained $E(t)$ defined as

$$\begin{aligned} E(t) &= \text{energy gained while foraging} \\ &\quad - \text{energy spent while foraging and traveling.} \end{aligned}$$

(This means that $E(t) = f(t) - r(t+\tau)$ where r is the rate of energy spent per unit time and τ is the fixed travel time).

Assume as before that the energy gained by foraging for a time t in the food patch is $f(t) = E_{max}t/(k+t)$.

(a) Find the amount of time t spent foraging that maximizes $E(t)$.

(b) Indicate a condition of the form $k < \boxed{?}$ that is required for existence of this critical point.

Cannon ball

. A cannonball is shot vertically upwards from the ground with initial velocity $v_0 = 15\text{m/sec}$. It is determined that the height of the ball, y (in meters), as a function of the time, t (in sec) is given by

$$y = v_0 t - 4.9t^2$$

Determine the following:

(a) The time at which the cannonball reaches its highest point,

(b) The velocity and acceleration of the cannonball at $t = 0.5$ s, and $t = 1.5$ s.

(c) The time at which the cannonball hits the ground.

Review problem (1)

(A) Sketch $f(x)$ given the following information: $\underline{f(1) = 0}$

x	-1	0	1	2	3	4
$f'(x)$	0	$+$	0	$+$	0	$-$

Review Problem (2)

(2) Sketch $f(x)$ given the following information

x	0	1	2	3	4
f	0		-1		1
f'		0	0	0	0
f''		$-$	$+$	$-$	$+$

Review Problem (3)

(3) Consider the function $f(x) = x^3 - 3ax^2 + 3x$

(a) For what range of values of a does this function have no critical points?

(b) For what value of a does $f(x)$ have an inflection point at $x = 0$?